Garth van der Kamp

PERIODIC FLOW OF GROUNDWATER

A Systematic Study of Wave Propagation under Confined, Semiconfined and Unconfined Flow Conditions

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TABLE OF CONTENTS

ABSTRACT
A comprehensive theory is developed for the propagation of waves through thin aquifers, including the effects of storage in the aquitards and vertical flow in the aquifers. For various cases of confined, semiconfined and unconfined flow solutions are given whose ranges of applicability are quantitatively defined.
Data reduction methods for periodic motions are discussed, with special reference to tidal effects.
Empirical results from a number of different sites are presented and compared with the theoretical predictions. It is shown on the basis of the data that the theory is for a large part valid and its application to actual situations is illustrated. For cases of confined and unconfined flow some, perhaps significant, discrepancies between theory and observation are found

FOREWORD

The present work started out as an attempt to apply measurements of tidal fluctuations in coastal aquifers to the determination of the geohydrological characteristics of the formations, using existing theory. It became apparent after a while however that the applicability of the various available analytical models and the assumptions on which they were based had not received enough attention, so that it was difficult to say whether discrepancies between theory and observation were due to the inappropriateness of the models, or to weaknesses in the basis theoretical assumptions, or simply to irregularities in hydrostratigraphy. One could of course simply leave the discrepancies at that, or explain them away in a rather ad hoc manner, but such an approach was not likely to lead to better understanding. Thus the main theme of the work gradually shifted to the development and empirical evaluation of a more comprehensive theory for the propagation of waves through aquifers. Through this approach my own understanding of groundwater flow has been considerably improved; I hope now that it may of some use to others.

During the process of coming to recognize and then grappling with these problems the aid and encouragement of many different people has been invaluable to me. In particular I would like to express my thanks to Dr. H. Lazreg, Dr. M. L. Parsons, Mr. D. H. Lennox and especially Dr. P. A. Carr, of the Hydrology Research Division in Ottawa. Among the many others who contributed through discussions and in the collection and analysis of data my thanks go especially to the following: Mr. M. Sylvestre, Dr. B. Dousse, Mr. A. J. Roebert, Mr. R. A. Schuurmans, Dr. J. Wesseling, and my Free University colleagues Mr. J. J. de Vries and Mr. Th. J. Beukeboom.

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1. INTRODUCTION

1.1. *The problem and the approach to it*

Variations in the water level of open bodies of water such as rivers or the sea generally induce corresponding variations of groundwater pressure in underlying or adjacent waterbearing formations. The propagation of these fluctuations landinwards through the formations can be viewed as a particular case of time-dependent groundwater flow. A study of this propagation can lead to an understanding of the phenomena itself, and can yield information on the hydraulic characteristics of the formations and on the nature of time-dependent flow of groundwater.

Particularly interesting is the special case of the horizontal propagation of periodic fluctuations of hydraulic potential emanating from a long straight shoreline. This case can be treated analytically in a fairly straightforward manner, and is in fact often encountered in practice, the most common example being that of tidal fluctuations in coastal aquifers. It is this problem, involving the transmission of periodic fluctuations, which has been selected for detailed study, both theoretical and empirical.

As mentioned above the propagation of periodic fluctuations is a particular case of time-dependent groundwater flow. Now most theoretical treatments of such flow are in effect based on two important assumptions, namely that the rate of flow is proportional to the gradient of hydraulic potential (Darcy's law), and that changes of storage of water are directly proportional to changes of hydraulic potential. This second assumption is commonly assumed to hold both for elastic storage within the saturated zone and for storage at the water table. To make analysis feasible a number of simplifying assumptions must be introduced, involving the structure and hydraulic characteristics of the formations, and the nature of the flow. Common examples of such assumptions are for instance: that the formations are homogeneous and isotropic; that the formations may be classified as either aquifers or aquitards (the aquifers being highly conductive as compared to the aquitards); that elastic storage within the aquitards is negligible; that the effect of vertical flow within the aquifers is negligible; or that the flow is either confined, semiconfined, or unconfined (i.e. completely, partially, or not at all isolated from the water table). The question of when and where assumptions such as these are valid has been receiving increasing attention in recent years, especially with regard to the problem of radial flow to a pumped well (Hantush, 1960; Neuman and Witherspoon, 1972; Wolff and

Papadopulos, 1972; Streltsova, 1972; Herrerra and Rodante, 1973). The problem with the introduction of assumptions of this sort is that although they may well be valid for most cases, adequate arguments establishing their validity are usually lacking, and as a result serious errors can be and have occasionally been made in the analysis and interpretation of groundwater flows.

The theoretical analysis of the propagation of periodic fluctuations that is given in the present work can be viewed in the light of the above discussion. The general approach is to derive solutions for periodic flow based on Darcy's law and the linear proportionality of changes of water storage and hydraulic potential. In addition the assumption of homogeneity and horizontal isotropy of the formations is introduced, but as far as possible it is attempted to avoid introducing at the outset any other assumptions involving the geohydrological characteristics of the formations or the nature of the flow. Further restrictions to various particular cases and solutions follow in the course of the analysis. These include a general restriction to flow in aquitards and thin aquifers, and others involving the flow type, and the importance of elastic storage, water table storage, and vertical flow in the aquifers. Because the analysis is kept general at the outset the range of applicability of these various solutions can be defined in terms of characteristic numbers, which are found as part of the results of the analysis. Thus for instance a quantitative definition will be given of the terms "aquifer" and "aquitard", "thick" and "thin", "confined" and "unconfined". The net result then of this general approach is a series of equations describing the transmission of periodic fluctuations for various special situations, equations whose ranges of applicability are delimited by exactly defined criteria. In addition it turns out that some of the equations previously derived by other authors must be modified because various factors, neglected in their derivation, are in fact not negligible.

The special value of such equations with clearly delimited ranges of applicability is that they may be applied with good confidence to the appropriate actual situations. A comparison of the theoretical and the empirical results can then serve as a check on the validity of the basic assumptions (Darcy's law, proportionality of storage and potential, homogeneity of the formations). Then, if it turns out that the basic assumptions allow for an adequate description of the flow, application of the equations can yield useful information on the hydraulic characteristics of the formations, and can allow prediction of potential fluctuations.

In connection with the evaluation of the theory on the basis of empirical results data from a number of different sites in the Netherlands and Canada are presented and analyzed. These consist both of data on the propagation of periodic fluctuations and of data on the geohydrological characteristics of the formations obtained by other means such as pumping tests. A comparison of the empirical results with the theoretical predictions allows an evaluation of the theory for cases of confined, semiconfined and unconfined flow, and also serves as an example of how the theory can be applied for the determination of the geohydrological characteristics of the formations. It should be noted that measurements of periodic fluctuations usually do not include data on the total amount of flow and therefore cannot yield as much information about the characteristics of the formations as pumping tests do.

1.2. *Historical sketch*

The discussion above presents the subject of this work and the approach that is used in terms of the present-day status of the theory for time-dependent groundwater flow. However it is also interesting to view the present work as part of a historical process of increasing understanding of groundwater flow in general and of periodic flow in particular. A series of authors have published theoretical and empirical results concerned with the periodic flow of groundwater. Their analytic treatment of the problem has sometimes been more complete and has taken more factors into account then the contemporary treatment of problems such as radial flow by other authors, probably because the problem of periodic flow is much simpler to treat analytically. Although the analysis that is given in this work is not directly dependent on previous treatments of the problem, it is yet indebted to them, for some of the empirical results that were obtained, and more importantly, for the problems that were delineated. A description of the problem of periodic flow would therefore not be complete without mention of at least some of the more significant results that have been obtained to date.

Probably the first detailed analytical treatment of the problem of periodic fluctuations is the thesis of Steggewentz (1933), in which a theory is presented for the propagation of periodic fluctuations through unconfined or semiconfined aquifers. Steggewentz assumed that storage of water through compression effects is negligible and therefore his theory is only applicable for cases where water table storage is dominant. The model he used falls short in some respects, but recognized the importance of vertical flow near the water table for unconfined flow – a point which has only recently gained general recognition (Streltsova, 1972, and others). Equations similar to those of Steggewentz were derived by Ernst (1962, p. 126) on the basis of Boulton's (1954) model of delayed yield from storage at the water table. Jacob (1940, 1950) considered the problem of tidal fluctuations in a completely confined aquifer, taking into account the compressibility of the water and the solid medium. He derived expressions for the propagation of the fluctuations through the inland part of the aquifer. Bosch (1951) presented an extension of Jacob's theory by including the effect of leakage to the water table. He did not take into account the effect of compressibility of the confining layer and of movements of the water table. The theory of Bosch was in turn extended by Wesseling (1959) to take into account the vertical displacements of the water table. Wesseling's analysis indicates that for semiconfined flow Bosch's assumption of a stationary water table is justified, and also indicates that loading effects due to water storage at the water table are negligible.

The effect of the compressibility of the confining layers was first considered by Edelman (1953) and later on, independently, by Ernst (1962, p. 138). Both authors arrived at the same expression for the propagation of periodic fluctuations through a confined aquifer, including the effect of flow and storage in the confining layer.

On the whole these various authors have not given a detailed discussion of the applicability of the equations which they derived. As a result the relationships between the various solutions is not clear, and at times the equations have been improperly applied to practical situations.

While the theory for periodic flow was being developed, empirical work was also done, usually with the two related purposes of checking the validity of the theory, and of investigating to what extent the measurement of periodic fluctuations could yield information on the hydraulic characteristics of the formations.

Ferris (1951) measured the potential fluctuations due to daily fluctuations in the state of a river. He applied Jacob's theory to calculate the ratio of transmissibility to storage coefficient. Timmers (1955) measured the transmission through an aquifer of fluctuations due to seasonal changes in the stage of a river. He analyzed the data on the basis of the theory of Steggewentz (1933) and obtained values for the hydraulic characteristics of the aquifer that were in good agreement with the expected values. A number of carefully carried out investigations of tidal groundwater fluctuations were reported by Wesseling (1960), van Eyden, Kuper and Santema (1963), and de Ridder and Wit (1965). Their results, mostly analyzed on the basis of the theory of Bosch (1951), agree at best approximately with the results of pump tests carried out in the vicinity. Carr (1971) applied Fourier analysis to show that the tidal fluctuations in wells can be separated into a number of sinusoidal components and that the propagation of each of these components was at least approximately consistent with Jacob's (1950) theory for confined flow.

In general it may be said of these and other empirical results that although agreement with theory is good in some cases, it is not good so in other cases, and on the whole no satisfactory explanations have been advanced for the discrepancies.

In view of the preceeding historical sketch the purpose of the present work can be stated once more. It is firstly to develop a theory for the propagation of periodic fluctuations which integrates and extends the previously developed theory, and secondly to present empirical data on the basis of which the validity and usefulness of the theory can be evaluated.

1.3. *Summary of chapter contents*

The material is arranged in a natural sequence leading from development of the theory through reduction and presentation of the data to a final comparison of theoretical and observational results. Some of the chapters may however be read independently of the others.

In chapter 2 the basic equations for time-dependent flow of groundwater are presented and applied to the case of simply periodic flow. It is shown that for thin aquifers there is a special solution which does not require detailed knowledge of the boundary conditions.

In chapter 3 this special solution for thin aquifers is derived and analyzed in detail for confined, semiconfined, and unconfined flow. In addition the problems of boundary conditions, entry resistances, and reflection at an internal boundary are discussed.

In chapter 4 methods of reducing and analyzing data for periodic fluctuations are discussed. This chapter may be read independently of the others.

In chapter 5 empirical data from various sites is presented and summarized. This chapter is included for completeness, and is not essential to an understanding of the other chapters.

In chapter 6 the theoretical results are summarized and evaluated on the basis of the empirical data. For the reader whose main interest is the application of the theory for periodic flow this chapter probably gives sufficient information.

2 – THEORY FOR PERIODIC FLOW OF GROUNDWATER

2.1. – *Introduction*

The transmission of periodic fluctuations in hydraulic potential through aquifers has been analyzed by various authors as summarized in the introductory chapter. Some of the more significant of these treatments are those of Steggewentz (1933), Jacob (1950), Wesseling (1959), and Ernst (1962). These and other authors have solved the problem of periodic flow for particular cases of confined, semiconfined, and unconfined aquifers, but their analysis in general does not clearly define the range of validity of the various solutions or the relationship between them. In view of this consideration the main purpose of this and the following chapter can be stated. It is to present a theory for the propagation of periodic fluctuations through a system of horizontal aquitards and thin aquifers. This analysis is based on Darcy's law, and the assumption that changes of storage of water within the aquifer or at the water table are directly proportional to the changes of hydraulic potential. It will yield some new or modified results, but its most important result is a more general theory of which most of the previously derived results are special cases. In this manner the relationship between the various particular solutions can be made clear, and quantitative criteria can be derived which define their range of validity. In this chapter the general equations governing the flow are derived. The application of these equations for various particular cases of confined, semiconfined, and unconfined flow is left for the following chapter.

2.2. – *The hydrostratigraphic model*

The hydrostratigraphic model for which a general solution will be sought is that of a three-layer system bounded above by the water table as schematized in Figure 2–1. The bottom layer is assumed to be thick and of low permeability, so that it isolates the system from the effects of flow at deeper levels. This model is chosen because it is the simplest configuration which can represent both confined and unconfined conditions.

The layers 1, 2, and 3 have thicknesses $D_j = z_{j-1} - z_j$, horizontal hydraulic conductivities K_j, vertical hydraulic conductivities K_j', and specific storage coefficients S_j', where the subscript "j" stands for the number of the layer and may therefore take the values 1, 2, or 3. (The specific storage coefficient S_j' must be distinguished from the storage coefficient for the entire thickness of the layer, S_j.) These quantities are assumed to be constant throughout, i.e. the layers are assumed to be homogeneous and of constant thickness. Except for the limitation that layer 3 be a thick aquitard, these quantities may take a wide range of values. Thus layer 1 may for instance be an aquifer or an aquitard, or may be very thin, or absent altogether. In the course of the analysis one further restriction will be introduced namely that if a layer is an aquifer it must be thin. The terms "thick", "thin", "aquifer", and "aquitard" will be more precisely defined in the course of the analysis.

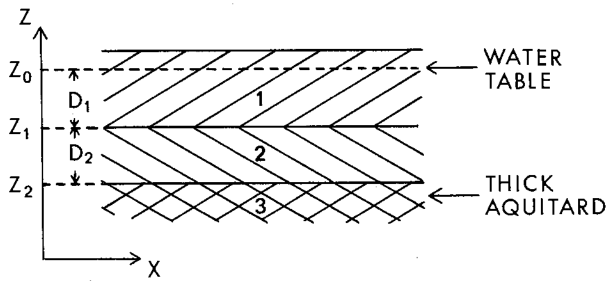

Figure 2–1 The hydrostratigraphic model

The average height of the water table, or, more accurately, of the top of the zone of saturation, is taken at $z = z_0$. The specific yield, or storage coefficient at the water table, is denoted by S_0 and will be assumed constant.

2.3. – *The basic equations*

2.3.1. – *The flow equation*

The basic differential equation describing the time-dependent flow of groundwater in a saturated porous medium has been derived by Cooper (1966), Verruijt (1969), and others. In the following analysis horizontal-vertical anisotropy is taken into account, and the treatment is restricted to the case of two-dimensional flow. The equation can then be written:

$$K_j \frac{\partial^2 h_j}{\partial x^2} + K_j' \frac{\partial^2 h_j}{\partial z^2} = S_j' \frac{\partial h_j}{\partial t}. \qquad 2\text{–}1$$

where h_j is the hydraulic potential, and the subscript "j" refers to the number of the layer as above. This form of the equation depends on the assumption that Darcy's law holds and that the elastic storage of water varies linearly with the hydraulic potential. A detailed discussion of these and other assumptions involved in the derivation of the equation has been given by the authors mentioned above.

2.3.2. – *Flow and storage at the water table*

Vertical gradients of the hydraulic potential in the saturated zone directly below the water table induce flow to or from the water table (in accordance with common practice the term "water table" is used here in place of the more accurate term "top of the zone of saturation".) If the water table is not held constant through drainage systems or the like, such flow results in the filling or emptying of pores and a consequent vertical movement of the water table. The processes involved in such movements of the water table are complex. For the present purpose it will be assumed in accordance with common practice (see for instance Childs, 1969) that the rate of vertical movement of the water table is directly proportional to the vertical flow within the saturated zone at the water table. If the instantaneous height of the water table is denoted by z_w, this linear relationship may be written as :

$$-K_1' \frac{\partial h_1}{\partial z}\Big|_{z=z_w} = S_0 \frac{\partial h_1}{\partial t}\Big|_{z=z_w} \qquad 2\text{–}2$$

where S_0 is the specific yield or coefficient of storage at the water table, and use has been made of the fact that the rate of change of potential at the water table is equal to the rate of change of height of the water table. This relationship can be written more simply if the displacements of the water table are very small, for then the left and right hand sides of equation 2–2 may be expanded as follows:

$$-K_1' \frac{\partial h_1}{\partial z}\Big|_{z=zw} = -K_1' \frac{\partial h_1}{\partial z}\Big|_{z=z_0} - K_1' \frac{\partial^2 h_1}{\partial z^2}\Big|_{z=z_0} (z_w - z_0) \qquad 2\text{–}3$$

$$S_0 \frac{\partial h_1}{\partial t}\Big|_{z=z_w} = S_0 \frac{\partial h_1}{\partial t}\Big|_{z=z_0} + S_0 \frac{\partial^2 h_1}{\partial z \partial t}\Big|_{z=z_0} (z_w - z_0) \qquad 2\text{–}4$$

The terms on the right side of equations 2–3 and 2–4, involving the displacement of the water tabel $(z_w - z_0)$ are negligible if this displacement is very small. For such cases the equation for flow and storage at the water table may be written:

$$-K_1' \frac{\partial h_1}{\partial z}\Big|_{z=z_0} = S_0 \frac{\partial h_1}{\partial t}\Big|_{z=z_0} \qquad 2\text{–}5$$

Equation 2–5 can be considered more generally to describe also those cases where the water table is fixed, as by drainage systems, or where there is no storage at the water table. In accordance with the previously implied definition of the specific yield S_0 as the ratio between the flow to the water table and the rate of change of potential at the water table these two cases can be described through equation 2–5 by putting S_0 equal to infinity or zero respectively. In this manner these special cases of flow and storage at the water table are included in the general analysis.

2.3.3. – *Boundary conditions*

The boundary conditions on the horizontal planes between the layers basically involve continuity of flow and potential. The stipulation that layer 3 be a thick aquitard can be expressed by the condition that the fluctuations in potential become negligibly small at depths far below z_1. With inclusion of the boundary condition at z_0 expressed by equation 2–5, the boundary conditions at the horizontal boundaries can be written:

$$z=z_0 \quad : \quad -K_1' \frac{\partial h_1}{\partial z} = S_0 \frac{\partial h_1}{\partial t} \qquad \text{2–6a}$$

$$z=z_1 \quad : \quad h_1 = h_2 \qquad \text{2–6b}$$

$$z=z_1 \quad : \quad K_1' \frac{\partial h_1}{\partial z} = K_2' \frac{\partial h_2}{\partial z} \qquad \text{2–6c}$$

$$z=z_2 \quad : \quad h_2 = h_3 \qquad \text{2–6d}$$

$$z=z_2 \quad : \quad K_2' \frac{\partial h_2}{\partial z} = K_3' \frac{\partial h_3}{\partial z} \qquad \text{2–6e}$$

$$z \rightarrow -\infty \quad h_3 = 0 \qquad \text{2–6f}$$

Since the analysis is to be restricted to the case of two-dimensional flow, the remaining requirement for a complete description of the problem is a specification of the potentials at the two ends of the region of interest. These may in general be written as:

$$h_j \; (x = x_1, z, t) = u(z, t), \qquad \text{2-7a}$$

$$h_j \; (x = x_2, z, t) = v(z, t), \qquad \text{2–7b}$$

where the functions u (z, t) and v (z, t) are periodic in time. For many cases one of the positions x_1 or x_2 can be taken at infinity and the potential at that point can than be required to be vanishingly small. A general solution of the problem described by equations 2–1, 2–6, and 2–7 will not be attempted here. As will be shown, a general form satisfying equations 2–1 and 2–6 can be found in a fairly straightforward manner. A special solution will then be selected which is applicable for the case where the detailed forms of the functions u (z, t) and v (z, t) in equations 2–7 do not appear in the solution. Observations indicate that such a solution is indeed applicable for many of the cases encountered in practice. The criteria for the cases where the complete solution is closely approximated by this special solution will be estimated on the basis of a general solution for the simple but representative special case where layer 2 is an aquifer and layers 1 and 3 are completely impermeable.

2.3.4. – *Linearity*

The equations governing the flow are all linear in the potential h_j (x, z, t). This property of the equations implies that different types of flow such as flow to a pumped well, seasonal flows, and various periodic flows occur superimposed on each other but independently. (This conclusion is of course only valid if the total displacements of the water table are small compared to the thickness of layer 1.) Each flow type can therefore be treated separately. In particular, a complicated periodic motion can be considered as consisting of a number of sinusoidal components of various frequencies each of which can be separately considered. The analysis for periodic flow of this and the following chapter will be restricted to the consideration of a single sinusoidal motion, and the potential functions h_j (x, z, t) will refer only to the changes in potential due to one such motion. Because of the linearity of the equations the "background" potential, whether constant, or varying with time or place, need not be considered.

One other useful consequence of the linearity of the equations is that it allows the use of a complex potential function in the form $h_j = Q_j + i\,R_j$ where i is the square root of minus one, and the real quantities Q_j and R_j are respectively the real and imaginary parts of h_j. From the fact that the equations are linear it follows that if h_j satisfies the equations then so do Q_j and R_j separately. The actual hydraulic potential (a real function) may then be defined as either the real or imaginary part of the complex potential h_j. The advantage of such a complex representation is that it allows a compact representation of a periodic function and its derivates.

2.4. – *Solution*

A solution for equations 2–1 and 2–6 will now be sought for periodic flow. One important assumption is introduced at the outset, namely that of the separability of the vertical and horizontal potential distributions, expressed as:

$$h_j(x, z, t) = g_j\,(z, t)\; f_j\,(x, t) \qquad 2\text{–}8$$

Substitution of this form of h_j in the basic differential equation (equation 2–1) gives:

$$\frac{K_j'\,\dfrac{\partial^2 g_j}{\partial z^2} - S_j'\,\dfrac{\partial g_j}{\partial t}}{g_j} = \frac{-K_j\,\dfrac{\partial^2 f_j}{\partial +^2} + S_j'\,\dfrac{\partial f_j}{\partial t}}{f_j} \qquad 2\text{–}9$$

Since one side of this equation is independent of z, and the other side independent of x, they must both be a function of t only, which for convenience at a later stage is written as $K_j'\, q_j^2\,(t)$. Then equation 2–9 gives:

$$K_j\,\frac{\partial^2 f_j}{\partial x^2} + K_j' q_j^2 f_j = S_j'\,\frac{\partial f_j}{\partial t} \qquad 2\text{–}10$$

$$K_j'\,\frac{\partial^2 g_j}{\partial z^2} - K_j'\, q_j^{\,2} g_j = S_j'\,\frac{\partial g_j}{\partial t} \qquad 2\text{–}11$$

The assumption of separability thus leads to separate equations for the horizontal and vertical potential distributions, related through the function $q_j^2(t)$.

A further simplification is possible for two instances, namely those of flow in thin layers, and of periodic flow in layers of any thickness. If a layer is thin the vertical potential distribution across it will be always in a state of quasi-equilibrium, dependent only on the vertical flow through the upper and lower surfaces, but independent of the changes of potential within the layer. In terms of equation 2–11 this condition can be expressed as:

$$\left|S_j'\,\frac{\partial g_j}{\partial t}\right| << \left|K_j'\,\frac{\partial^2 g_j}{\partial z^2}\right| \qquad 2\text{–}12$$

This case of flow in thin aquifers is included here because it will yield a general equation for such flow which will be useful for interpreting the empirical results for periodic flow.

If the flow is simply periodic, i.e. purely sinusoidal in time, the function $g_j\,(z, t)$ can always be written as a function of z only. Then condition 2–12 holds automatically whether the layer is thick or thin. Moreover it then follows from equation 2–9 and the subsequent reasoning that for such periodic flow g_j becomes independent of time. Thus for any kind of flow in thin layers, or for simply periodic flow in layers of any thickness, condition 2–12 holds, and equation 2–11 then becomes:

$$\frac{\partial^2 g_j}{\partial z^2} - q_j^{\,2}\, g_j = 0 \qquad 2\text{–}13$$

Equations 2–10 and 2–13 combined with equations 2–6 can now be solved for general forms of the horizontal and vertical potential distributions as will be shown in the next sections.

2.4.2. – *General forms of the horizontal and vertical potential distributions*

Equations 2–6b and 2–6d, involving the continuity of the potential on the boundaries $z=z_1$, and $z=z_2$, give:

$$g_1(z_1)\, f_1(x,t) = g_2(z_1)\, f_2(x,t) \qquad 2\text{–}14$$

$$g_2(z_2)\, f_2(x,t) = g_3(z_2)\, f_3(x,t) \qquad 2\text{–}15$$

It follows that f_1, f_2, and f_3 are proportional to each other and in fact it is possible to put $g_1(z_1) = g_2(z_1)$ and $g_3(z_2) = g_2(z_2)$ so that:

$$f_1(x,t) = f_2(x,t) = f_3(x,t) = f(x,t) \qquad 2\text{–}16$$

This horizontal potential distribution $f(x,t)$ is governed by equation 2–10. For purely sinusoidally varying periodic flow the general solution for this equation can be written (in complex representation):

$$f(x,t) = (A_1 e^{px} + A_2 e^{-px})e^{i\omega t} \qquad 2\text{–}17$$

where A_1, A_2, and p are complex numbers, p can be written as:

$$p = n + im, \qquad 2\text{–}18$$

where n and m are real numbers, and n can be defined to be always positive.

Equation 2–13, governing the vertical potential distribution, has a general solution of the form:

$$g_j(z) = B_{1j} e^{q_j z} + B_{2j} e^{-q_j z} \qquad 2\text{–}19$$

With the boundary conditions given by equations 2–6 this general form of $g_j(z)$ can be further specified. (Because the solutions are still very general they are necessarily complicated in form, and therefore for

the sake of a compact presentation the steps in the analysis which involve only algebraic manipulation are not given in detail.)
As was done for p the real part of q_j can always be chosen to be positive. It then follows from equation 2–6f that:

$$g_3(z) = B_{13}e^{q_3 z}, \qquad 2\text{–}20$$

which can be written:

$$g_3(z) = g_3(z_2)e^{q_3(z-z_2)} \qquad 2\text{–}21$$

With equation 2–16, equations 2–6d and 2–6e, involving the continuity of potential and vertical flow at $z=z_2$ give:

$$B_{12}e^{q_2 z_2} + B_{22}e^{-q_2 z_2} = g_3(z_2) \qquad 2\text{–}22$$

$$B_{12}e^{q_2 z_2} - B_{22}e^{-q_2 z_2} = \frac{q_3 K_3'}{q_2 K_2'}\, g_3(z_2) \qquad 2\text{–}23$$

Solution of equations 2–22 and 2–23 for B_{12} and B_{22}, and substitution of the results in the general form of $g_j(z)$ given by equation 2–19 gives:

$$g_2(z) = g_3(z_2)\left\{\cosh[q_2(z-z_2)] + \frac{q_3 K_3'}{q_2 K_2'}\sinh[q_2(z-z_2)]\right\} \qquad 2\text{–}24$$

Equations 2–6b and 2–6c, involving continuity of flow, and potential at $z=z_1$, together with equation 2–16 yield (with $D_2 = z_1 - z_2$):

$$B_{11}e^{q_1 z_1} + B_{21}e^{-q_1 z_1} = g_3(z_2)[\cosh(q_2 D_2) + \frac{q_3 K_3'}{q_2 K_2'}\sinh(q_2 D_2)] \qquad 2\text{–}25$$

$$B_{11}e^{q_1 z_1} - B_{21}e^{-q_1 z_1} = g_3(z_2)\frac{q_2 K_2'}{q_1 K_1'}\,[\sinh(q_2 D_2) + \frac{q_3 K_3'}{q_2 K_2'}\cosh(q_2 D_2)] \qquad 2\text{–}26$$

Solution of these equations for B_{11} and B_{21}, and substitution into the general form of $g_j(z)$ yields finally:

$$g_1(z) = g_3(z_2)\left\{\cosh[q_1(z-z_1)]\,[\cosh(q_2 D_2) + \frac{q_3 K_3'}{q_2 K_2'}\sinh(q_2 D_2)] + \right.$$

$$\left. + \sinh[q_1(z-z_1)]\,[\frac{q_3 K_3'}{q_1 K_1'}\cosh(q_2 D_2) + \frac{q_2 K_2'}{q_1 K_1'}\sinh(q_2 D_2)]\right\} \qquad 2\text{–}27$$

General forms of the horizontal potential distribution f(x,t), and the vertical potential distributions $g_1(z)$, $g_2(z)$, and $g_3(z)$ have now been found and are given by equations 2–17, 2–27, 2–24, and 2–21 respectively. It should be noted that the derivation of the form of the vertical potential distributions has not involved the assumption that the flow is periodic.

2.4.3. – *The general solution*

The solutions for p, q_1, q_2, and q_3 can be found through application of equation 2–10 and of equation 2–6a for flow and storage at the water table. Substitution of the general form $g_1(z)$ as given by equation 2–27 into equation 2–6a gives (after division by $g_3(z_2)$, and rearrangement of the terms):

$$[\cosh(q_2 D_2) + \frac{q_3 K_3'}{q_2 K_2'} \sinh(q_2 D_2)][q_1 K_1' \sinh(q_1 D_1) f + S_0 \cosh(q_1 D_1)\frac{\partial f}{\partial t}] +$$
$$+ [q_2 K_2' \sinh(q_2 D_2) + q_3 K_3' \cosh(q_2 D_2)][\cosh(q_1 D_1) f + \frac{S_0}{q_1 K_1'} \sinh(q_1 D_1)\frac{\partial f}{\partial t}] = 0 \qquad 2\text{–}28$$

This equation can be viewed as giving a relation between the q_j's and p (implicit in the function $f(x,t)$). It can be reduced to simpler forms for a number of special cases as will be shown in the next chapter. Substitution of the general form of $f(x,t)$ as given by equation 2–17 into equation 2–10 gives:

$$K_j' q_j^2 = i\omega S_j' - K_j p^2 \qquad 2\text{–}29$$

Equations 2–28 and 2–29 together form a system of four equations in four unknowns q_1, q_2, q_3, and p. However, the hyperbolic functions which appear in equation 2–28 are periodic with a period of $2\pi i$. Therefore there may be many sets of values q_1, q_2, and q_3 which satisfy equation 2–28. For each of these sets equation 2–29 will also yield a different value of p. Thus the general solution can be written:

$$h_j(x,z,t) = \sum_r g_j^{(r)}(z) f^{(r)}(x,t) \qquad 2\text{–}30$$

where the superscript (r) denotes the various potential distributions corresponding to the various sets of q_j and p which satisfy equations 2–28 and 2–29. A complete solution would be obtained if the sum in equation 2–30 satisfies the boundary conditions expressed by equations 2–7. Such a complete solution for all possible cases will not be attempted here. For many practical cases of periodic flow one special solution of equations

2–28 and 2–29 can be selected as giving a good approximation to the complete solution. This special solution and the criteria for its applicability will be considered next.

2.4.4. – *The general solution for a single layer with no vertical flow through its upper or lower surfaces*

From the general three-layer case a special solution will be selected and analyzed in detail. For the purpose of illustrating the selection of this solution the case will be considered here where layer 2 is an aquifer and layers 1 and 3 are completely impermeable ($K_1' = K_3' = 0$). If the aquifer is supposed to extend indefinitely in the x-direction, the boundary conditions (equations 2–7) may, for the simply periodic case, be written:

$$h_2(x = x_1, z, t) = E(z)e^{i\omega t} \qquad 2\text{–}31a$$

$$h_2(x = \infty, z, t) \text{ remains finite} \qquad 2\text{–}31b$$

With $K_1' = K_3' = 0$ equation 2–28, multiplied through by $q_1 K_1' q_2 K_2'$ gives:

$$q_2{}^2 K_2{}'^2 \sinh(q_2 D_2) S_0 \frac{\partial f(x,t)}{\partial t} = 0 \qquad 2\text{–}32$$

Since K_2', S_0, D_2, and $\partial f/\partial t$ are not zero, this equation has the solutions:

$$q_2 D_2 = i\pi r \qquad 2\text{–}33a$$

where $\quad r = 0, \pm 1, \pm 2 \ldots.$ $\qquad$ 2–33b

and then equation 2–29 gives:

$$p_r{}^2 = i\omega \frac{S_2'}{K_2} + \frac{\pi^2 r^2}{K_2 D_2 c_2} \qquad 2\text{–}34$$

where $c_2 = D_2/K_2'$ is the vertical hydraulic resistance of layer 2. By equation 2–24, the vertical potential distributions have the form:

$$g_2^{(r)}(z) = g_2^{(r)}(z_2)\cosh[i\pi r(z - z_2)/D_2] \qquad 2\text{–}35a$$

$$= g_2^{(r)}(z_2)\cos[\pi r(z - z_2)/D_2] \qquad 2\text{–}35b$$

At $x = x_1$, $g_2(z)$ can be taken equal to E (z), Thus, with equation 2–30 and 2–35b:

$$\sum_{r=0}^{\infty} g_2^{(r)}(z_2)\cos[\pi r(z - z_2)/D_2] = E(z) \qquad 2\text{–}36$$

(Since cos (–x) = cos(x) the negative values of r need not be included). The sum on the left hand side of this equation can be viewed as a Fourier expansion of E(z), and in fact equation 2–36 leads to:

$$g_2^{(o)}(z_2) = \frac{1}{D_2}\int_{z_2}^{z_1} E(z)dz \qquad 2\text{–}37a$$

and for $r \neq 0$

$$g_2^{(r)}(z_2) = \frac{2}{D_2}\int_{z_2}^{z_1} E(z)\cos[\pi r(z - z_2)/D_2]dz \qquad 2\text{–}37b$$

From equation 2–31b it follows that the horizontal potential distributions have the form (based on equations 2–17 and 2–18):

$$f^{(r)}(x,t) = A_2^{(r)} \exp.\left\{-n_r(x-x_1) + i[\omega t - m_r(x-x_1)]\right\} \qquad 2\text{–}38$$

The complete solution has the form:

$$h_2(x,z,t) = \sum_{r=0}^{\infty} g_2^{(r)}(z)f^{(r)}(x,t) \qquad 2\text{–}39$$

where $g_2^{(r)}(z)$ and $f^{(r)}(x,t)$ are given by equations 2–33 through 2–38. Here again the summation may be restricted to nonnegative values of r, since $p = n + im$ is determined by r^2 (equation 2–34).
Thus, for this special case of a single isolated layer (in fact a completely confined aquifer) a general solution has been found. The assumption of separability of the variables, introduced through equation 2–8 is justified at least for this special case since it permits a complete solution.
If certain conditions are satisfied the special solution for $r = 0$, i.e. $q_2 D_2 = 0$, closely approximates the complete solution, as will be shown next. From equation 2–28 it follows that the larger n_r, the more rapidly the corresponding part of the total fluctuation is damped out with distance away from the boundary $x = x_1$. Now, equation 2–34 gives:

$$n_o^2 = \omega S_2'/2K_2 \qquad 2\text{–}40a$$

$$n_r^2(r \neq 0) = \frac{\pi^2 r^2}{2K_2 D_2 c_2} + \left[\left(\frac{\omega S_2'}{2K_2}\right)^2 + \left(\frac{\pi^2 r^2}{2K_2 D_2 c_2}\right)^2\right]^{\frac{1}{2}} \qquad 2\text{–}40b$$

It follows that n_o is much smaller than n_r $(r \neq 0)$ if:

$$\omega S_2 c_2 \ll \pi^2 \qquad 2\text{--}41$$

(Here $S_2 = S_2{}'D_2$ as previously defined). Thus if condition 2–41 holds, the fluctuation corresponding to r = 0 is much less quickly damped out with distance away from the boundary then the fluctuations corresponding to $r \neq 0$.
The solution corresponding to r = 0 closely approximates the complete solution in the region where:

$$|g_2^{(o)}(z)f^{(o)}(x,t)| \gg |\sum_{r=1}^{\infty} g_2^{(r)}(z)f^{(r)}(x,t)| \qquad 2\text{--}42$$

Equation 2–36 implies that $A_2^{(r)} = 1$. The amplitudes $g_2^{(r)}(z_2)$ can be further determined through equation 2–37. The function E(z) appearing in the boundary condition and in equation 2–37 can be written:

$$E(z) = \overline{E} + e(z). \qquad 2\text{--}43$$

where $\overline{E}$ is the average value of E(z) over the thickness of layer 2, and where the average value of e(z) is zero. Equations 2–37 then give:

$$g_2^{(0)}(z_2) = \frac{1}{D_2}\int_{z_2}^{z_1} E(z)dz = \overline{E} \qquad 2\text{--}44a$$

and for $r \geq 1$:

$$g_2^{(r)}(z_2) = \frac{2}{D_2}\int_{z_2}^{z_1} e(z)\cos[\pi r(z - z_2)/D_2]dz \qquad 2\text{--}44b$$

It follows that if the vertical potential differences at the boundary $x = x_1$ are small, i.e.

$$|e(z)| < |\overline{E}| \qquad 2\text{--}45$$

then (with $r \geq 1$):

$$|g_2^{(r)}(z_2)| < |g_2^{(o)}(z_2)| \qquad 2\text{--}46$$

(If condition 2–45 holds more strongly i.e. a "$\ll$" sign substituted for the "<" sign, than so does condition 2–46).

Considering for the moment only the solutions corresponding to r = 0, and r = 1, condition 2–42 can be written:

$$|g_2^{(o)}(z_2)e^{-n_0(x-x_1)}| >> |g_2^{(1)}(z_2)e^{-n_1(x-x_1)}| \qquad 2\text{–}47$$

If condition 2–46 holds for $g_2^{(1)}(z_2)$ in particular, condition 2–47 is satisfied if:

$$e^{(n_1-n_0)(x-x_1)} >> 1 \qquad 2\text{–}48$$

If condition 2–41 holds, then n_1 is much greater than n_0 and is in fact approximately equal to $\pi(K_2D_2c_2)^{-\frac{1}{2}}$. Then with $e^{\pi} = 23$ reckoned as much greater than unity, condition 2–48 can be written:

$$x - x_1 > (K_2D_2c_2)^{\frac{1}{2}} \qquad 2\text{–}49$$

Thus if the aquifer is sufficiently thin so that condition 2–41 holds, and if the vertical potential differences at the boundary are small (conditions 2–45 and 2–46 hold), then at points sufficiently far away from the boundary (satisfying condition 2–49), the special solution for r = 0 closely approximates the general solution. In fact the equation for the potential distribution can then be written in the form:

$$h_2(x,z,t) = A_2e^{-px}e^{i\omega t} \qquad 2\text{–}50a$$

where $A_2 = g_2^{(0)}(z_2)$ and p is given by:

$$p^2 = i\omega S_2'/K_2 \qquad 2\text{–}50b$$

Of course, if the vertical gradients at the boundary are very small (i.e. conditions 2–45 and 2–46 hold more stringently), then condition 2–49 can be relaxed accordingly. In practice the vertical gradients at the boundary are usually known to be small, either from direct measurement or from experience of similar cases. Also, the more "thin" the aquifer (i.e. the more stringently condition 2–41 holds), the less significant the vertical gradients are likely to be.

If the layer is isotropic $(K_2' = K_2)_1$ then the quantity $(K_2D_2c_2)^{\frac{1}{2}}$ in condition 2–49 equals D_2, and condition 2–49 then gives a result similar to that of Hantush (1962) and others for flow to a partially penetrating well in an isotropic aquifer. Hantush shows that for such flow the vertical potential gradients in the aquifer due to the effects of partial penetration of the well become negligible at a few aquifer thicknesses from the well. Condition 2–49 indicates that not the aquifer thickness D_2, but the quantity $(K_2 D_2 c_2)^{\frac{1}{2}} = D_2(K_2/K_2')^{\frac{1}{2}}$ is the decisive factor.

It should be noted that when the complete solution is approximated by the solution for $q_2 D_2 = 0$, as expressed by equations 2–50, then the amplitude is determined by the average of the potential at the boundary. For the rest the solution is then independent of the detailed form of E(z), the vertical potential distribution at the boundary. Thus this solution can be applied even when no detailed information is available for the potential differences at the boundary. In particular the horizontal propagation of the fluctuations through the layer is then described by the propagation parameter $p = n + im$ as given by equation 2–50b, which depends only on the fact that the fluctuation is sinuosoidal with angular frequency ω.

2.5. *The solution for points far away from the boundary and thin aquifers*

2.5.1. – *Selection of the solution*

In the previous section a general solution has been found for the special case when layers 1 and 3 are completely impermeable. The problem remains of finding solutions for the more general three-layer case. A complete solution for this case will not be attempted; but the solution for the one-layer case indicates that it may be possible to find a good approximation to the complete solution for points far away from the boundary if the aquifers are thin.

In fact it was found that for one layer case the solution with $q_2 D_2 = 0$ $(r = 0)$ dominates at points far away from the boundary satisfying condition 2–40.

$$x - x_1 > (K_2 D_2 c_2)^{\frac{1}{2}} \qquad 2\text{–}51$$

if the vertical potential gradients at the boundary are small and if the aquifer is sufficiently thin, i.e. if the aquifer satisfies condition 2–41, which may be written:

$$\omega S_2 c_2 < 1 \qquad 2\text{–}52$$

The horizontal propagation parameter $p^{(0)}$ which was found as part of the solution for $q_2 D_2 = 0$ is independent of the detailed form of the boundary condition and can therefore be applied in practical situations even when there is no detailed information of the boundary conditions.

Now this result for the one-layer case indicates that similar solutions may exist for the three-layer case if the aquifers (layers 1 or 2 or both) are "thin". When the three layer case is restricted to that of a thin aquifer bounded above and below by thick aquitards, then it is very similar to the one-layer case, and in fact there is no clear break between them since the one-layer case is then approached more and more closely as the aquitards become less and less conductive. When the flow in layer 2 is not isolated from the water table, i.e. when layer 1 is thin or absent altogether, then there is vertical flow through the upper surface, and the solution for the one-layer case found in the previous section does not apply, even approximately. However, it may be safely assumed, that for such cases also, the effect of vertical potential differences at the boundary should be small at points far away from the boundary if the aquifers (layer 1 or 2 or both) are thin. In general then, it should be possible to find a solution for the three-layer case applicable to the region far away from the boundary. This solution, corresponding to the solution $q_2 D_2 = 0$ for one confined layer, can be sought through the condition that if layer "j" is an aquifer then:

$$|q_j D_j| << 1 \qquad 2\text{–}53$$

This solution should include the solution $q_2 D_2 = 0$ as a special case when $K_1' = K_3' = 0$. It may also be expected that the other solutions for the three-layer case not satisfying condition 2–53, will then correspond to the solutions with $q_2 D_2 = i\pi r$ (r a nonzero integer) for the one-layer case– solutions which were found to be negligible at points far away from the boundary.

A rigorous proof of these suppositions will not be given here. It will be assumed that solutions obtained through condition 2–53 do indeed give a good approximation of the complete solution for points far away from the boundary when the aquifers are thin. Empirical results may help to corroborate the validity of this assumption for practical cases. Further analysis will be restricted to the derivation of solutions satisfying condition 2–53 for the aquifers; From here on the symbols p and q_j will be used to refer to this special solution only, and the other possible solutions will not be further considered.

2.5.2. – *Definitions of the terms "aquifer", "aquitard", "thick" and "thin"*

The terms "aquitard" and "aquifer" are relative and can only be applied when there is a large difference of hydraulic conductivity between the layers. A layer is an aquitard if the horizontal flow through it is negligible compared to that in the much more conductive aquifer. The term aquitard can for the present purposes be restricted to those layers in which the flow is largely or completely vertical. In terms of equation 2–29 this restriction can be conveniently expressed through the condition:

$$|K_j' q_j^2| >> |K_j p^2| \qquad 2\text{–}54$$

for then equation 2–29 reduces to:

$$q_j^2 \text{ (aquitard)} = i\omega S_j'/K_j' \qquad 2\text{–}55$$

Condition 2–54 will be taken as the defining condition for an aquitard because it is consistent with the earlier definitions, and permits the simple solution for the number q_j expressed by equation 2–55. Condition 2–54 may also be interpreted as implying that the horizontal distance through which the fluctuations are transmitted in the aquifer (proportional to p^{-1}) is much larger than the vertical distance through which the fluctuations are transmitted within the aquitard (proportional to q_j^{-1}).
With this result for q_j the terms "thin" and "thick" as applied to aquitards can be further specified. If for an aquitard:

$$|q_j D_j| < 1 \qquad 2\text{–}56$$

then a significant part of the fluctuation is transmitted vertically through the layer. With equation 2–55 this condition can be replaced by:

$$\omega S_j c_j < 1 \qquad 2\text{–}57$$

Here the square of the quantity $q_j D_j$ has been used because condition 2–57, which can be taken as the definition of a "thin" aquitard, then becomes identical to condition 2–52 defining a "thin" aquifer. Condition 2–57 may thus be taken as the general definition of a "thin" layer for the purposes of the present analysis.
Similarly the condition:

$$|q_j D_j| >> 1 \qquad 2\text{–}58$$

implies that the fluctuations are not transmitted vertically through the layer to a significant extent. (Here use is made of the fact that for aquitards the real and imaginary parts of q_j are equal, as shown by equation 2–55). In fact, condition 2–58 can be replaced by the less restrictive condition (analogous with condition 2–57):

$$\omega S_j c_j >> 1 \qquad 2\text{–}59$$

This condition will be taken as the definition of a "thick" aquitard for the present purposes.
The terms "thin" and "thick", as applied to aquitards, have now been defined relative to the vertical penetration distance of the fluctuations. For this reason not only the thickness of the layers is involved in these definitions, but also the hydraulic characteristics of the layers, and the frequency of the fluctuations.
The condition that layer 3 be a thick aquitard, which was introduced at the outset, can be quantitatively expressed through conditions 2–54 and 2–59.
The fluctuations do not penetrate horizontally through the aquitards to any significant extent. Therefore, if the effects of the vertical potential gradients at the boundary are negligible in the aquifers they will certainly be so in the aquitards. The special solution obtained through condition 2–53 thus remains applicable for points far away from the boundary if the effect of flow in the aquitards above and below the aquifers is considered. In fact equation 2–55 already gives the corresponding solution for q_j of the aquitards.

2.5.3. *The general form of the special solution*

If a solution for the flow in aquitards and thin aquifers satisfying condition 2–53 can be found it will have the general form expressed by equations 2–8, 2–17, and 2–19:

$$h_j(x,z,t) = g_j(z)f(x,t) \qquad 2\text{–}60\text{a}$$

$$= (B_{1j}e^{q_j z} + B_{2j}e^{-q_j z})(A_1 e^{px} + A_2 e^{-px})e^{i\omega t} \qquad 2\text{–}60\text{b}$$

As will be shown in the next chapter such a solution does indeed exist. It thus turns out that the assumption of separability of the variables as expressed through equations 2–8 (or 2–60a) is applicable for the flow in aquitards and thin aquifers. This theoretical result may be viewed as being due to the fact that the flow is either largely horizontal (in the aquifers), or largely vertical (in the aquitards).

Through equations 2–28 and 2–29 solutions for p and q_j can be found without reference to the potentials at the boundary other than that they are periodic in time. The main purpose of the analysis, which is to find equations for the horizontal propagation of the fluctuations, will thus be obtained. Various particular solutions for p and q_j are discussed in detail in the next chapter.

3 – SOLUTIONS FOR CONFINED, SEMICONFINED, AND UNCONFINED FLOW IN THIN AQUIFERS

3.1 – *Introduction*

3.1.1. – *Purpose*

In the previous chapter equations have been derived for periodic flow in a three-layer system, such as illustrated in Figure 3–1. A general solution for these equations has not been given, but it was found that if the aquifers are thin then for points far away from the boundary a particular solution can be found which closely approximates the complete solution. No rigorous proof of this conclusion has been given, but arguments which support it have been presented on the basis of a general solution for the case of one isolated layer with no flow through its upper or lower surfaces.

In this chapter these special solutions, applicable for points far away from the boundaries, and thin aquifers, will be derived and analyzed in detail for various special cases of confined, semiconfined, and unconfined periodic flow. Where feasible explicit solutions will be given, with special emphasis on the equations describing the horizontal propagation of the periodic fluctuations. The conditions describing the range of validity of the solutions will be given as completely as possible, but for a detailed analysis of some of these conditions the reader will be referred back to the previous chapter. The approach is similar to that of Hantush (1960) for radial flow to a pumped well, except that in the present case the effect of vertical flow within the aquifer is also taken into account. The problem of radial periodic flow (see Williams et al., 1970), and of periodic flow in thick aquifers (Carrier and Munk, 1952; Diprima, 1958) will not be considered.

The main purpose of this chapter is to derive equations for the propagation of periodic fluctuations whose range of validity is carefully circumscribed, so that they may be applied with confidence to the appropriate practical situations. The theoretical predictions which this analysis yields, when compared with empirical results, can give a trustworthy indication of the extent to which the basic theory that is used can indeed describe the flows encountered in practice.

3.1.2. – *The hydrostratigraphic models and the general equations*

The hydrostratigraphic model for which the theory has been developed is described in the previous chapter, and is reproduced in figure 3–1. Layer 3 is a thick aquitard which acts as a base isolating the flow in layers 1 and 2 from flow at greater depths.

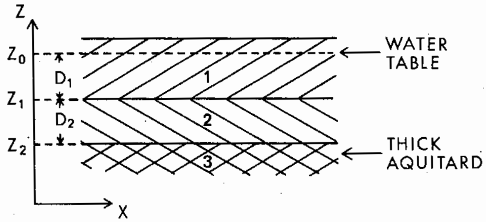

Figure 3–1 The hydrostratigraphic model

For simply periodic flow the solutions for the potential distributions have the form (equations 2–60):

$$h_j(x,z,t) = g_j(z)f(x,t) \qquad 3\text{–}1a$$

$$= (B_{1j}e^{q_j z} + B_{2j}e^{-q_j z})(A_1 e^{px} + A_2 e^{-px})e^{i\omega t} \qquad 3\text{–}1b$$

where h_j is the hydraulic potential in layer j. The vertical potential distributions $g_j(z)$ are given more exactly in equations 2–21, 2–24, and 2–27, which incorporate the solutions for the quantities B_{1j} and B_{2j}. The equations governing the quantities q_j and p have been given in chapter 2 (equations 2–28 and 2–29). They can be written:

$$K_j p^2 + K_j' q_j^2 = i\omega S_j' \qquad 3\text{–}2$$

$$[\coth(q_2 D_2) + \frac{q_3 K_3'}{q_2 K_2'}]\,[K_1' q_1 f + S_0 \coth(q_1 D_1)\frac{\partial f}{\partial t}] +$$

$$+ [q_2 K_2' + q_3 K_3' \coth(q_2 D_2)]\,[\coth(q_1 D_1) f + \frac{S_0}{q_1 K_1'}\frac{\partial f}{\partial t}] = 0 \qquad 3\text{–}3$$

Together they form a system of four equations in the four unknowns q_1, q_2, q_3, and p (implicit in f), and a complete solution is in principle

possible. (For the meaning of the symbols used in these and the following equations the reader is referred to chapter 2 or to the list of repeatedly used symbols given in the appendix.)
However, since the function coth (q_jD_j) is periodic with period πi, there may be many sets of q_1, q_2, and q_3 which satisfy equation 3–3, and therefore many solutions in the form of equation 3–1. The general solution is then some linear combination of all these particular solutions. As has been argued in the previous chapter, the solution derived through the condition that for the aquifers:

$$|q_jD_j| << 1 \qquad 3\text{–}4$$

gives a good approximation to the general solution for points far away from the boundaries if the aquifers are thin and if the vertical potential differences at the boundaries are not large. The region far away from the boundary at say $x = x_1$ is described by the condition (2–51):

$$x - x_1 > (K_jD_jc_j)^{\frac{1}{2}} \qquad 3\text{–}5$$

where layer "j" denotes an aquifer and x increases away from the boundary. This condition can be relaxed if the vertical potential differences at the boundary are very small. A layer "j", whether an aquifer or an aquitard, has been defined as thin if (condition 2–57):

$$\omega S_jc_j < 1 \qquad 3\text{–}6$$

For a more detailed discussion of these conditions and their derivations the reader is referred to chapter 2.
Solutions of equations 3–2 and 3–3 will be sought subject to condition 3–4. The q_jD_j found in this manner are only valid solutions if they do indeed satisfy condition 3–4, and in fact this requirement leads to a restriction for the range of applicability of the solutions. Conditions 3–5 and 3–6 give additional restrictions.
The analysis will only be carried out for two special cases, namely: (a) layer 1 is a thick aquitard and layer 2 is a thin aquifer (confined flow); (b) layers 1 and 2 are both thin, and flow in layer 3 is negligible (semiconfined and unconfined flow above an impermeable base). These cases, for which the layers are always either thick or thin, together with condition 3–4, allow the introduction of some simplifying approximations. Although many restrictions have by now been introduced on the range of applicability of the solutions, they do in fact cover a

wide range of practically occurring situations, and provide a conceptual framework for the analysis and understanding of more complicated cases.
The main concern of the analysis is to find explicit equations relating the propagation parameters q_j and p to the geohydrological characteristics of the layers. These equations can be found through equations 3–2 and 3–3 together with condition 3–4 without reference to the boundary conditions other than that the fluctuation at the boundary is periodic in time. They are only applicable for points sufficiently far away from the boundary, but since they require no detailed knowledge of the potential differences at the boundary they are nevertheless useful for the many cases where only data for the propagation of the fluctuations are available or can be obtained. Special attention will be given to the equations for p, because this quantity can be easily and reliably determined by field measurements. Some attention will also be given to the quantities A_1 and A_2 appearing in equation 3–1b, which are determined by the boundary conditions.

3.1.3 – *The approximations*

As indicated above, the analysis will only be carried out subjection to condition 3–4 for cases where the aquifers are thin (i.e. satisfy condition 3–6) and the aquitards are thin or thick.
The terms "aquifer" and "aquitard" were defined in chapter 2 through condition 2–54. For the aquitards equation 2–55 holds, which states:

$$q_j^2 \text{ (aquitards)} = i\omega S_j'/K_j' \qquad 3\text{–}7$$

With this equation condition 2–54 defining an aquitard can be written:

$$\omega S_j'/K_j' \text{ (aquitards)} >> |p^2| \qquad 3\text{–}8$$

This condition cannot be further specified until solutions for p have been found, but it is useful in this form because the magnitude of p can usually be estimated fairly well.
Condition 2–57, defining a "thin" aquitard, can be written (with use of equation 3–7):

$$|q_j^2 D_j^2| < 1 \qquad 3\text{–}9$$

When this condition holds the following approximations can be used:

$$\sinh(q_jD_j) \cong q_jD_j\,(1 + q_j^2D_j^2/6) \qquad 3\text{–}10a$$

$$\cosh(q_jD_j) \cong 1 + q_j^2D_j^2/2 \qquad 3\text{–}10b$$

$$\coth(q_jD_j) \cong (1 + q_j^2D_j^2/3)\,(q_jD_j)^{-1} \qquad 3\text{–}10c$$

Now condition 3–9 is a less restrictive version of condition 3–4, by which the solution for points far away from the boundary is selected. Since solutions based on condition 3–4 can also be sought through the approximations given in equations 3–10, such solutions would be identical to those based on condition 3–9. Therefore condition 3–9 will replace condition 3–4 in the following analysis.
Condition 2–59, defining a "thick" aquitard, can be written:

$$|q_j^2D_j^2| >> 1 \qquad 3\text{–}11$$

When this condition is satisfied, and with the real part of q_j defined to be positive as before, the following approximations can be used (utilizing the fact expressed by equation 3–7 that for aquitards the real and imaginary parts of q_j are equal):

$$\sinh(q_jD_j) \cong \cosh(q_jD_j) \cong \tfrac{1}{2}\,e^{q_jD_j} \qquad 3\text{–}12a$$

$$\coth(q_jD_j) \cong 1 \qquad 3\text{–}12b$$

The restriction to thin or thick aquitards and solutions satisfying condition 3–4 can be expressed through application of condition 3–9 to the aquifers, and condition 3–9 or 3–11 to the aquitards. The approximations given in equations 3–10 and 3–12 will be used to find the corresponding solutions of equations 3–2 and 3–3.

3.2. – *Confined periodic flow*

3.2.1 – *The solution for confined flow*

When layer 2 is an aquifer, the flow in this layer may be defined as being confined when it is isolated from the water table. This situation occurs when layer 1 is a thick aquitard as defined by condition 3–11. In addition it will be required that layer 2 be a thin aquifer and that layer 3 be a thick aquitard. A solution to equations 3–2 and 3–3 for this case is thus to be sought subject to conditions 3–9 for the aquifer and condition 3–11 for the aquitards:

$$| q_2^{\,2} D_2^{\,2} | < 1 \qquad 3\text{–}13a$$

$$| q_1^{\,2} D_1^{\,2} | >> 1 \qquad 3\text{–}13b$$

$$| q_3^{\,2} D_3^{\,2} | >> 1 \qquad 3\text{–}13c$$

Since layers 1 and 3 are specified to be aquitards satisfying condition 3–8, equation 3–7 gives:

$$q_1^{\,2} = i\omega S_1'/K_1' \qquad 3\text{–}14a$$

$$q_3^{\,2} = i\omega S_3'/K_3' \qquad 3\text{–}14b$$

The condition that the aquifer, layer 2, be thin, can be expressed through condition 3–6:

$$\omega S_2 c_2 < 1 \qquad 3\text{–}15a$$

With equations 3–14, conditions 3–13b and 3–13c can be written:

$$\omega S_1 c_1 < 1 \qquad 3\text{–}15b$$

$$\omega S_3 c_3 < 1 \qquad 3\text{–}15c$$

The symbols S_j and c_j are defined as follows:

$$S_j = S_j' D_j \qquad 3\text{–}16a$$

is the elastic storage coeefficient of layer j, and

$$c_j = D_j / K_j' \qquad 3\text{–}16b$$

is the vertical hydraulic resistance of layer j.
Conditions 3–13b and 3–13c imply that the approximation given by equation 3–12b is applicable for layers 1 and 3. With this approximation, equation 3–3 reduces to:

$$[\coth(q_2D_2) + \frac{q_3K_3'}{q_2K_2'} + \frac{q_2K_2'}{q_1K_1'} +$$

$$+ \frac{q_3K_3'}{q_1K_1'} \coth(q_2D_2)] \, (K_1'q_1 f + S_0\frac{\partial f}{\partial t}) = 0 \qquad 3\text{–}17$$

Consideration of equation 3–14a shows that the factor $q_1K_1'f + S_0\frac{\partial f}{\partial t}$ takes nonzero values. Therefore equation 3–17 implies that:

$$q_2K_2' \coth q_2D_2 \, (q_1K_1' + q_3K_3') + q_2^{\,2} K_2'^2 + q_1K_1'q_3K_3' = 0 \quad 3\text{–}18$$

Condition 3–13a implies that equation 3–10c applies to coth (q_2D_2). Equation 3–18 then yields finally:

$$q_2^{\;2} = \frac{-q_1K_1' - q_3K_3' - c_2q_1K_1'q_3K_3'}{K_2'D_2\,[\,1 + (q_1K_1' + q_3K_3')\,c_2/3]} \qquad 3\text{–}19$$

and with equation 3–2:

$$p^2 = i\omega\frac{S_2'}{K_2} + \frac{q_1K_1' + q_3K_3' + c_2q_1K_1'q_3K_3'}{K_2D_2\,[\,1 + (q_1K_1' + q_3K_3')\,c_2/3]} \qquad 3\text{–}20$$

Equations 3–14a, 3–14b, 3–19, and 3–20 together give the explicit solutions for q_1, q_2, q_3 and p in terms of the thicknesses and the hydraulic characteristics of the layers. For this case of confined flow therefore the problem of the transmission of periodic fluctuations is essentially solved.
With use of equation 3–19 the condition on q_2D_2 (condition 3–13a) can be written:

$$|c_2(q_1K_1' + q_3K_3' + c_2q_1K_1'q_3K_3')| <$$

$$|\,1 + (q_1K_1' + q_3K_3')\,c_2/3\,| \qquad 3\text{–}21$$

This condition is certainly satisfied if (using equations 3–14):

$$c_2\sqrt{\omega S_1'K_1'} < 1/3 \qquad 3\text{–}22a$$

$$c_2\sqrt{\omega S_3'K_3'} < 1/3 \qquad 3\text{–}22b$$

When these conditions are satisfied, than so is condition 3–9 by which the special solution for points far away from the boundary was selected. They imply that in the region where this solution is applicable the vertical potential gradients in the aquifer are small. Conditions 3–22 together with conditions 3–15 limit the applicability of the solution, even for points far away from the boundary.
The vertical potential distribution in layer 3 has already been given in equation 2–21 which states:

$$g_3(z) = g_2(z_2)\, e^{q_3 (z-z_2)} \qquad 3\text{–}23$$

Application of equation 3–18 to the general form of $g_1(z)$ given by equation 2–27 gives:

$$g_1(z) = g_2(z_1)\, e^{-q_1 (z-z_1)} \qquad 3\text{–}24$$

Thus the vertical potential distributions in the aquitards are exponential in form, decreasing with vertical distance away from the aquifer, layer 2, since the real parts of q_1 and q_3 are positive.
The vertical potential distribution within layer 2, $g_2(z)$, can be found through a combination of equations 2–24 and the approximations implied by equations 3–10a and 3–10b. $g_2(z)$ is a quadratic function of z, but its general form is complicated and is not given here. Suffice it to mention that for many practical instances of confined flow the vertical potential gradients within layer 2 are negligibly small, as will be discussed in the next section.

3.2.3. – *Special cases of confined periodic flow.*

The equations for confined periodic flow derived in the previous section treat the general case where the effects of storage and vertical flow in the aquifer and in the aquitard all may be significant. For most practical cases at least some of these effects are negligible, and the equations can then be further reduced.
For many commonly encountered cases conditions 3–22 are easily satisfied; in fact the more restrictive conditions:

$$c_2 \sqrt{\omega S_1' K_1'} \ll 1/3 \qquad 3\text{–}25a$$

$$c_2 \sqrt{\omega S_3' K_3'} \ll 1/3 \qquad 3\text{–}25b$$

often hold. With these conditions equation 3–19 for $q_2^{\,2}$ becomes:

$$q_2^{\,2} D_2^{\,2} = c_2 \left(\sqrt{i\omega S_1{}' K_1{}'} + \sqrt{i\omega S_3{}' K_3{}'} \right) \qquad 3\text{–}26$$

It follows from conditions 3–25 that then:

$$| q_2^{\,2} D_2^{\,2} | \ll 1 \qquad 3\text{–}27$$

When conditions 3–25 hold, the vertical potential distribution within the aquifer, $g_2(z)$, becomes a constant, and the vertical potential gradients within the aquifer are negligible in the region sufficiently far away from the boundary, where the solution obtained through these conditions is applicable. In fact, for such cases the vertical potential gradients in the aquifer are likely to be very small even near or at the boundary, and the solution is then also applicable near the boundary.
With conditions 3–25 holding, equation 3–20 for p^2 reduces to:

$$p^2 = i\omega \frac{S_2{}'}{K_2} + \frac{\sqrt{i\omega S_1{}' K_1{}'} + \sqrt{i\omega S_3{}' K_3{}'}}{K_2 D_2} \qquad 3\text{–}28$$

Separation of the real and imaginary parts of this equation gives:

$$n^2 - m^2 = \frac{1}{K_2 D_2} \left(\sqrt{\omega S_1{}' K_1{}' / 2} + \sqrt{\omega S_3{}' K_3{}' / 2} \right) \qquad 3\text{–}29a$$

$$2\,nm = \frac{\omega S_2}{K_2 D_2} + \frac{1}{K_2 D_2} \left(\sqrt{\omega S_1{}' K_1{}' / 2} + \sqrt{\omega S_3{}' K_3{}' / 2} \right) \qquad 3\text{–}29b$$

These are the equations describing the horizontal propagation of a simply periodic fluctuation for the case of confined flow when the effect of flow and storage in the aquitards is significant. Similar equations have been previously derived by T. Edelman (1953), and Ernst (1962).
When the flow in the aquitards is negligible, i.e. when:

$$\sqrt{\omega S_1{}' K_1{}' / 2} + \sqrt{\omega S_3{}' K_3{}' / 2} \ll \omega S_2 \qquad 3\text{–}30$$

equations 3–29 reduce to:

$$n^2 \approx m^2 \qquad 3\text{–}31a$$

$$2nm = \frac{\omega S_2}{K_2 D_2} \qquad 3\text{–}31b$$

These equations for completely confined periodic flow have been given by Jacob (1950) and others. The model for which they hold (flow in aquitards negligible) is frequently used in the analysis of periodic groundwater flow, but in fact an investigation of condition 3–30 indicates that in practice the effect of flow in the aquitards cannot usually be neglected. For these special cases the equations for the vertical propagation of the fluctuations into the aquitards (equations 3–14, 3–23, and 3–24) remain valid without further reduction.

Condition 3–4, by which the solution for points far away from the boundary is selected, was derived in chapter 2 through consideration of the general solution for the one-layer case. One consequence of the reasoning used was that when the aquitards above and below layer 2 are impermeable then the solution obtained through condition 3–4 (or 3–9) should be identical to the special solution with $q_2 D_2 = 0$ for the one-layer case. Equations 3–31, applicable when the flow in the aquitards is negligible, confirm the validity of the reasoning of chapter 2, for they give the same result for n and m (or p = n+im) as was obtained for the one-layer case with $q_2 D_2 = 0$ (equation 2–34).

3.3. – *Semiconfined and unconfined periodic flow*

3.3.1. – *Limiting conditions*

For the present purpose it is convenient to define the flow as semiconfined if the position of the water table plays a significant role in determining the flow pattern, but the movement of the water table is negligible. Similarly the flow may be defined as unconfined if it is largely or completely governed by both the position and the movement of the water table. Both these types of flow can be included in one analysis if layers 1 and 2 are both assumed to be thin, satisfying condition 3–6:

$$\omega S_1 c_1 < 1 \qquad \text{3–23a}$$

$$\omega S_2 c_2 < 1 \qquad \text{3–32b}$$

The solution for points far away from the boundary can then be selected through condition 3–9:

$$| q_1{}^2 D_1{}^2 | < 1 \qquad \text{3–33a}$$

$$| q_2{}^2 D_2{}^2 | < 1 \qquad \text{3–33b}$$

Layer 3 must be an aquitard compared to one or both of the layers above it, in other words condition 3–8 applied to layer 3 must be satisfied:

$$\omega S_3'/K_3 >> |p^2| \qquad 3\text{–}34$$

Layer 3 is then an aquitard, and the general condition that it be thick can be written:

$$\omega S_3 c_3 >> 1 \qquad 3\text{–}35$$

To avoid excessively complicated equations it will be assumed that the effects of flow in the bottom aquitard (layer 3) are negligible. Equation 3–3 shows that this assumption is valid if:

$$|q_3 K_3'| << |q_2 K_2' \coth q_2 D_2| \qquad 3\text{–}36a$$

$$|q_3 K_3'| << |q_2 K_2' \text{ tank } q_2 D_2| \qquad 3\text{–}36b$$

From the limitations on the magnitude of $q_1 D_1$ and $q_2 D_2$ expressed by conditions 3–33 and the corresponding approximations given by equation 3–10c it follows that conditions 3–36 are always fairly well satisfied if:

$$\sqrt{\omega S_3' K_3'} << \frac{1}{c_2} |q_2{}^2 D_2{}^2| \qquad 3\text{–}37$$

This condition will be further analyzed when an expression for $q_2 D_2$ has been obtained.

3.3.2. – *General equations for semiconfined and unconfined periodic flow*

In the following analysis the assumption of periodic flow will not be immediately introduced. Equation 3–3 holds for all types of flow in thin layers, and in fact, together with equation 2–10, it leads to a differential equation for semiconfined and unconfined flow in thin layers not restricted to periodic flow. This result is interesting enough to warrant inclusion in an analysis otherwise restricted to periodic flow.

When conditions 3–33 hold, the approximation expressed by equation 3–10c applies to $\coth(q_1 D_1)$ and $\coth(q_2 D_2)$. Equation 2–10, derived at the beginning of the previous chapter, may be written:

$$q_j{}^2 D_j{}^2 f = S_j c_j \frac{\partial f}{\partial f} - K_j D_j \frac{\partial^2 f}{\partial x^2} \qquad 3\text{–}38$$

With equations 3–10c and 3–38, and with neglect of flow in layer 3 and of second order terms, equation 3–3 becomes:

$$[K_1 D_1 + K_2 D_2] \frac{\partial^2 f}{\partial x^2} + S_0 [K_1 D_1 c_1 /3 + $$
$$+ K_2 D_2 (c_1 + c_2/3)] \frac{\partial^3 f}{\partial t \partial x^2} = [S_0 + S_1 + S_2] \frac{\partial f}{\partial t} +$$
$$+ S_0 [S_1 c_1 /3 + S_2 (c_1 + c_2/3)] \frac{\partial^2 f}{\partial t^2} \qquad 3\text{–}39$$

The assumption of periodic flow has not been used in the derivation of this equation, and it is therefore a general differential equation for semi-confined and unconfined flow in thin layers, subject to the restriction introduced through condition 3–9 that it is only applicable for points far away from the boundaries where the effects of the vertical potential differences at the boundaries are small. For nonperiodic flow the condition defining a layer as "thin" was given in chapter 2 (condition 2–12). It should also be mentioned that equation 3–39 is only valid when the total displacements of the water table are small compared to the saturated thickness of the layer within which it lies.

Equation 3–39 is similar in form to the equation derived by Boulton (1954, 1963) on the basis of a "delayed yield" model for unconfined flow. When for instance layer 1 is a very thin aquitard ($K_1 D_1 \ll K_2 D_2$, $S_1 \ll S_2$) equation 3–39 becomes:

$$K_2 D_2 \left[\frac{\partial^2 f}{\partial x^2} + S_0 (c_1 + c_2/3) \frac{\partial^3 f}{\partial t \partial x^2}\right] = (S_0 + S_2) \frac{\partial f}{\partial t} +$$
$$+ S_0 S_2 (c_1 + c_2/3) \frac{\partial^2 f}{\partial t^2} \qquad 3\text{–}40$$

which is identical with Boulton's equation if his "delayed yield index" is interpreted as being identical to $S_0(c_1 + c_2/3)$. It has recently been shown by Neuman (1972), Cooley (1972), and Streltsova (1972) that the "delayed yield" behavior of unconfined flow may be accounted for through consideration of the effect of vertical flow within the aquifer. The above analysis confirms this conclusion, for the terms involving $c_1/3$ and $(c_1 + c_2/3)$ represent the effect of vertical flow. The factor 1/3 in these terms expresses the fact that when the vertical flow is dispersed within the layer, most of this vertical flow is concentrated in the top or bottom of the layer.

The solution of equation 3–39 for simply periodic flow is straightforward. The function f(x,t) has the general form given in equations 3–1. With the following definitions:

$$a = S_0 + S_1 + S_2 \qquad \text{3–41a}$$

$$b = S_1 c_1/3 + S_2(c_1 + c_2/3) \qquad \text{3–41b}$$

$$d = K_1 D_1 c_1/3 + K_2 D_2 (c_1 + c_2/3) \qquad \text{3–41c}$$

$$e = K_1 D_1 + K_2 D_2 \qquad \text{3–41d}$$

the solution for p^2 is:

$$p^2 = \frac{i\omega a - \omega^2 S_0 b}{e + i\omega S_0 d} \qquad \text{3–42}$$

This equation for p is complicated in its generality, but nearly always reduces to simpler forms for particular cases, as will be shown in the next sections.
The equations for q_1 and q_2 are derived through equation 3–2. With the following definitions:

$$a_{ij} = \omega S_i c_j \qquad \text{3–43a}$$

$$a_i' = \omega S_i (c_1 + c_2/3) \qquad \text{3–43b}$$

the equations for q_1 and q_2 can be written:

$$q_1{}^2 D_1{}^2 = \\ = \frac{-K_1 D_1 [ia_{21}(1 + ia_0') + ia_{01}] + K_2 D_2 [ia_{11}(1 + ia_0')]}{K_1 D_1 (1 + ia_{01}/3) + K_2 D_2 (1 + ia_0')} \qquad \text{3–44a}$$

$$q_2{}^2 D_2{}^2 = \\ = \frac{K_1 D_1 [ia_{22}(1 + ia_{01}/3)] - K_2 D_2 [ia_{12}(1 + ia_{01}/3) + ia_{02}]}{K_1 D_1 (1 + ia_{01}/3) + K_2 D_2 (1 + ia_0')} \qquad \text{3–44b}$$

These equations are also complicated, but again reduce to simpler forms for many special cases. They have been included here because the vertical potential gradients are often significant for semiconfined and unconfined flow. The equations describing the vertical gradients can be found by a combination of equations 3–41 with the general form of $g_1(z)$ and $g_2(z)$ as given in equations 2–24 and 2–27, and with use of the approximations implied by equations 3–10. These, and conditions 3–33 will not be written out explicitly here because of their complicated form. They will be analyzed for some special cases in the next sections. Suffice it to mention that when conditions 3–33 hold $g_2(z)$ will in general be a quadratic function of z, and $g_1(z)$ will in general be a cubic function of z.

3.3.3. – *The special case when layer 1 is an aquitard*

Equations 3–42 and 3–44 give the propagation parameters p, q_1, and q_2, in terms of the thicknesses and the hydraulic characteristics of the layers. These equations can be reduced to simpler forms for various special cases. One general condition which nearly always holds is that:

$$S_0 \gg S_1 + S_2 \qquad 3\text{–}45$$

This condition expresses the assumption that the storage at the water table is much larger than the elastic storage. It will be used in the analysis of this section.

One interesting special case with many possibilities of application is that where layer 1 is an aquitard and satisfies the condition:

$$K_1 D_1 \ll K_2 D_2 \qquad 3\text{–}46$$

Then equations 3–44 for $q_1 D_1$ and $q_2 D_2$ reduce to:

$$q_1^2 D_1^2 = i\omega S_1 c_1 \qquad 3\text{–}47a$$

$$q_2^2 D_2^2 = -\frac{i\omega S_0 c_2 (1 + i\omega S_1 c_1/3)}{1 + i\omega S_0 c'} \qquad 3\text{–}47b$$

where $c' = c_1 + c_2/3$ 3–48

Equation 3–47a is identical to equation 3–7 for the q_j of aquitards. Conditions 3–33 now reduce (approximately) to:

$$\omega S_1 c_1 < 1 \qquad 3\text{–}49a$$

$$\omega S_0 c_2 < \sqrt{1 + \omega^2 S_0{}^2 c'^2} \qquad 3\text{–}49b$$

and condition 3–37 which expresses the condition that the flow in the base be negligible then becomes:

$$\omega S_0 >> (1 + \omega S_0 c_1)\sqrt{\omega S_3' K_3'} \qquad 3\text{–}50$$

With this condition that layer 1 be a thin aquitard, equation 3–42 for p^2 reduces to:

$$p^2 = \frac{i\omega S_0 [1 + i\omega S_1 c_1/3 + i\omega S_2 c']}{K_2 D_2 [1 + \omega^2 S_0{}^2 c'^2]} \qquad 3\text{–}51$$

Separation of the real and imaginary parts of equation 3–51 gives:

$$n^2 - m^2 = \frac{\omega^2 S_0{}^2 c'}{K_2 D_2 (1 + \omega^2 S_0{}^2 c'^2)} \qquad 3\text{–}52a$$

$$2nm = \frac{\omega S_0 [1 + \omega S_0 c' (\omega S_1 c_1/3 + \omega S_2 c')]}{K_2 D_2 (1 + \omega^2 S_0{}^2 c'^2)} \qquad 3\text{–}52b$$

Equations similar to these have been derived by Wesseling (1959), who neglected the vertical hydraulic resistance of layer 2 and the elastic storage in layer 1, but included the effects of loading due to storage of water at the water table (a negligible effect as it turns out).
Still further reduction is possible for many particular cases. A series of these can be listed as follows:

(a) *Apparently confined flow*

Suppose

$$\omega S_2 c' >> 1 \qquad 3\text{–}53a$$

then through condition 3–45:

$$\omega S_0 c' >>> 1 \qquad 3\text{–}53b$$

and equations 3–52 reduce to:

$$n^2 - m^2 = 0 \qquad 3\text{–}54a$$

$$2nm = \frac{\omega S_2}{K_2 D_2} \qquad 3\text{–}54b$$

These equations are identical to equations 3–31 for completely confined flow, and illustrate thereby the coninuity between confined and semiconfined flow when the overlying aquitard is thin and highly impermeable. Condition 3–53a can only be satisfied when S_2 is much greater than S_1, a rare situation in practice.

(b) *Semiconfined flow*
Suppose:

$$(\omega S_1 c_1/3 + \omega S_2 c')(\omega S_0 c') >> 1 \qquad 3\text{–}55a$$

and

$$\omega S_0 c' >> 1 \qquad 3\text{–}55b$$

then equations 3–52 become:

$$n^2 - m^2 = \frac{1}{K_2 D_2 c'} \qquad 3\text{–}56a$$

$$2nm = \frac{\omega S_1 c_1/3 + \omega S_2 c'}{K_2 D_2 c'} \qquad 3\text{–}56b$$

For this case the position, but not the movement, of the water table play a role in determining the flow. Equations similar to these but not including the elastic storage in the aquitard, have been presented by Bosch (1951).

(c) *Unconfined flow (with vertical potential gradients in the aquifer)*
Suppose:

$$\omega S_0 c' \; (\omega S_1 c_1/3 + \omega S_2 c') << 1 \qquad 3\text{–}57$$

Then equations 3–52 can be written:

$$n^2 - m^2 = \frac{\omega^2 S_0{}^2 c'}{K_2 D_2 \; (1 + \omega^2 S_0{}^2 c'^2)} \qquad 3\text{–}58a$$

$$2nm = \frac{\omega S_0}{K_2 D_2 \, (1 + \omega^2 S_0{}^2 c'^2)} \qquad 3\text{–}58b$$

As can be seen from these equations, the flow in this case is governed by the position and movement of the water table, and the effects of elastic storage within the layers is negligible. Equations similar to these have been derived by Steggewentz (1933) who assumed however that only the vertical hydraulic resistance of the covering aquitard was important. Ernst (1962) derived similar equations using Boulton's (1954) "delayed yield" model, not a surprising result in view of the similarity of equations 3–39 and 3–40 to Boulton's equation.
For this case the effects of vertical potential gradients within the aquifer are usually not negligible, especially when the overlying aquitard is not present. In the next section a detailed analysis will be given of these gradients when the flow is unconfined.

(d) *Unconfined flow (effect of vertical potential gradients negligible)*
When:

$$\omega S_0 c' \ll 1 \qquad 3\text{–}59$$

then, with condition 3–45, equations 3–52 give:

$$n^2 - m^2 = 0 \qquad 3\text{–}60a$$

$$2nm = \frac{\omega S_0}{K_2 D_2} \qquad 3\text{–}60b$$

For this case, which applies only for unconfined flow in very thin aquifers, the effect of vertical potential gradients within the aquifer is negligible. Equations equivalent to equations 3–60 have been derived by Werner and Noren (1951) on the basis of Dupuit's assumption of a constant hydraulic gradient in any vertical.

3.3.4. – *The vertical potential distribution for unconfined flow*

As was mentioned in the previous section, the vertical potential gradients within the aquifer are usually not negligible for unconfined flow. The reason for this lies in the fact that, in contrast to confined and semiconfined flow, for unconfined flow all the storage takes place at the water table, i.e. at the top of the aquifer, rather than throughout it as for elastic storage.

The flow is unconfined when condition 3–57 hold. Then equation 3–47b for q_2 becomes:

$$q_2{}^2 D_2{}^2 = -\frac{i\omega S_0 c_2 + \omega^2 S_0{}^2 c_2 c'}{1 + \omega^2 S_0{}^2 c'^2} \qquad 3\text{–}61$$

Substitution in the general form of the vertical potential distribution $g_2(z)$ as given in chapter 2 (equation 2–24) gives (with $q_3 K_3{}' = 0$, and the approximation implied by equation 3–10a):

$$g_2(z) = g_2(z_2)\,[1 + q_2{}^2\,(z-z_2)^2/2] \qquad 3\text{–}62$$

and with equation 3–61:

$$g_2(z) = g_2(z_2)\left[1 - \frac{(\omega^2 S_0{}^2 c_2 c' + i\omega S_0 c_2)\,(z-z_2)^2}{(1 + \omega^2 S_0{}^2 c'^2)\; 2D_2{}^2}\right] \qquad 3\text{–}63$$

In terms of amplitudes and phases this equation implies that going from the bottom to the top of the aquifer the amplitude decreases, and the phase lag increases. It should be noted that these equations are not valid if conditions 3–49, limiting the thickness of the layers, are not satisfied.

The vertical potential distribution in layer 1, $g_1(z)$ is approximately a cubic function of z when condition 3–33a holds (as consideration of equations 2–27 and 3–10 shows). If layer 1 is very thin, i.e. condition 3–32a holds more stringently, then $g_1(z)$ is approximately a linear function of z.

3.3.5 – *The special cases of no storage at, or no movement of the water table*

As was already mentioned in chapter 2, special cases of flow and storage at the water table can be included in the results of the general analysis, as long as S_0, the ratio of the amount of water stored to the potential change, at the water table, is constant.

When there is no storage at the water table, as occurs for instance when the zone of capillary rise reaches to the ground surface, S_0 can be taken as zero. Then equation 3–39 reduces to:

$$(K_1 D_1 + K_2 D_2)\frac{\partial^2 f}{\partial x^2} = (S_1 + S_2)\,\frac{\partial f}{\partial t} \qquad 3\text{–}64$$

This type of flow is equivalent to the completely confined case, as described by equations 3–31.

When the water table is held fixed, as by drainage, S_0 can be taken as very large or infinite, and integration of equation 3–39 gives:

$$(K_1 D_1 c_1/3 + K_2 D_2 c')\frac{\partial^2 f}{\partial x^2} = (S_1 c_1/3 + S_2 c')\frac{\partial f}{\partial t} + f \qquad 3\text{–}65$$

Here the integration constant has been put equal to zero since only changes of potential are being considered. For simply periodic flow the solution for p is:

$$p^2 = \frac{1 + i\omega(S_1 c_1/3 + S_2 c')}{K_1 D_1 c_1/3 + K_2 D_2 c'} \qquad 3\text{–}66$$

When layer 1 is an aquitard this equation gives:

$$n^2 - m^2 = \frac{1}{K_2 D_2 c'} \qquad 3\text{–}67a$$

$$2nm = \frac{\omega S_1 c_1/3 + \omega S_2 c'}{K_2 D_2 c'} \qquad 3\text{–}67b$$

These equations are identical to equations 3–56 for semiconfined flow when the water table is free to move. Their significance is somewhat different however, for in this case of a fixed water table equations 3–67 apply even when the frequency of the periodic fluctuation is very small or zero (steady flow). In fact when:

$$\omega S_1 c_1/3 + \omega S_2 c' << 1 \qquad 3\text{–}68$$

equations 3–67 reduce to:

$$n^2 = \frac{1}{K_2 D_2 c'} \qquad 3\text{–}69a$$

$$m = 0 \qquad 3\text{–}69b$$

and then the equation for the horizontal potential distribution f(x,t) becomes:

$$f(x,t) = (A_1 e^{x(K_2 D_2 c')^{-\frac{1}{2}}} + A_2 e^{-x(K_2 D_2 c')^{-\frac{1}{2}}})e^{i\omega t} \qquad 3\text{–}70$$

These equations, which apply even for steady flow, are equivalent to the "polder" equation derived by Mazure (1932). Mazure's equation and elaborations of it have been widely applied. The assumption of steady flow on which it is based is valid when condition 3–68 holds.

3.4 – *Horizontal flow and boundary effects*

3.4.1 – *Match to boundary conditions*

In the previous sections equations 3–2 and 3–3 have been solved for some special cases of simply periodic flow in a three-layer system consisting of thin aquifers and thick or thin aquitards. A general solution of the problem of simply periodic flow in such layers would also have to satisfy whatever boundary conditions might be imposed at the ends of the flow system. Such a complete solution has not been given, except in chapter 2 for the case of a single isolated layer with no flow through its upper and lower surfaces. It was argued in chapter 2 however that solutions which are obtained subject to condition 3–4 (or 3–9) should closely approximate the exact solution at points far away from the boundaries, and the analysis was carried through on that basis. For many cases with special boundary conditions these particular solutions do in fact represent the complete solution. The solution for points far away from the boundary, which has been obtained in the previous sections, has the general form (equations 3–1):

$$h_j(x,z,t) = g_j(z)\,(A_1 e^{px} + A_2 e^{-px})\, e^{i\omega t} \qquad 3\text{–}71$$

In the foregoing analysis the vertical potential distributions $g_j(z)$, and the horizontal propagation parameter p have been determined in terms of the thicknesses and the hydraulic characteristics of the layers. Except for some cases of unconfined flow, the function $g_j(z)$ nearly always reduces to a constant within thin aquifers. For some special cases the quantities A_1 and A_2 can be determined through the boundary conditions as shown below.

The potential fluctuations in the aquifer are usually induced by some external source such as a sea subject to tidal fluctuations, or a river with changing stage. In nearly all such cases there is an unknown entry resistance between the aquifer and the external source. Loading effects and complicated vertical flows also may play a role in the transmission of the fluctuations from the open water to the aquifer. In most cases therefore the potential fluctuation of the open water cannot be used directly as a boundary condition for the fluctuations in the aquifer.

Instead some point within the aquifer can be used, most conveniently taken at the observation point nearest the source of the fluctuation. If this point is far enough away from the shoreline to satisfy condition 3–5, then the actual vertical potential distribution will probably closely match that specified by the particular solution as in equation 3–71 above. With x being the distance from the (straight) shoreline, if the system is unbounded in the positive x-direction, the boundary conditions can be written:

$$h_j(x = x_0, z, t) = g_j(z) A_0 e^{i\omega t} \qquad 3\text{–}72a$$

$$h_j(x \longrightarrow \infty, z, t) \text{ remains finite,} \qquad 3\text{–}72b$$

where x_0 is the position of the observation point nearest the shoreline. With these boundary conditions the solution for points inland from x_0 is:

$$h_j(x,z,t) = g_j(z) A_0 e^{-n(x-x_0)} e^{i[\omega t - m(x-x_0)]} \qquad 3\text{–}73$$

Thus, when the boundary is taken sufficiently far inland the potential fluctuations can to a good approximation be described by the particular solutions which have been derived in the previous sections. When the aquifer is very thin, vertical potential differences in the aquifer are usually very small, and $g_j(z)$ then reduces to a constant for the aquifer.

For such cases the particular solution is applicable even near the source of the fluctuations.
Solutions similar to that of equation 3–73 but more complicated in form can be derived for cases of multiple boundaries, such as for flow in a long narrow island or land tongue.

3.4.2 – *The case of entry resistance*

In general the relationship between the external open water fluctuations and the potential fluctuations within the aquifer is complicated. However, a fairly straightforward analysis is possible for the case of flow through an entry resistance, a roughly vertical thin resisting layer, as illustrated in figure 3–2.

In figure 3–2 layer 2 is a thin aquifer, and layers 1 and 3 are aquitards. Layer 1 may be thick or thin or absent altogether. The entry resistance may be considered as vertical if the length of its projection on the x axis is much less than the horizontal distance through which the fluctuations penetrate the aquifer (which may be taken as equal to $1/n$). Similarly, loading effects will be negligible if the horizontal distance from the aquifer outcrop to the shoreline is much less than $1/n$, and if the entry resistance is small enough to allow passage of a considerable fraction of the fluctuation into the aquifer.

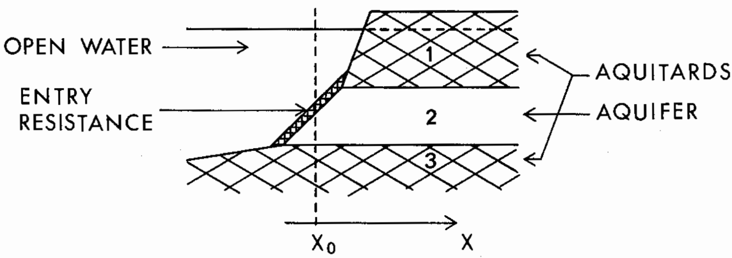

Figure 3–2 Aquifer separated from open water by an entry resistance

Let the potential fluctuation of the open water be represented by:

$$h_s = A_s e^{i\omega t} \qquad 3\text{–}74$$

and the potential within the aquifer, assuming that vertical potential differences are negligible, be represented by:

$$h_2(x = x_0, t) = A_0 e^{-p(x-x_0)} e^{i\omega t + 1_0} \qquad 3\text{–}75$$

where as A_s and A_0 are real, and 1_0 represents the phase change due to flow through the entry resistance. The horizontal flow through the entry resistance must be equal to the horizontal flow in the aquifer at $x = x_0$. If the hydraulic resistance of the entry resistance is denoted by C_e, this equality gives:

$$\frac{h_s - h_2(x = x_0)}{C_e} = -K_2 \left.\frac{\partial h_2}{\partial x}\right|_{x = x_0} \qquad 3\text{–}76$$

yielding:

$$h_2(x = x_0, t) = \frac{1}{1 + K_2 C_e p} A_s e^{i\omega t} \qquad 3\text{–}77$$

Separated into its real and imaginary parts equation 3–77 gives:

$$\left(\frac{A_0}{A_s}\right)^2 = \frac{1}{(1 + K_2 C_e n)^2 + (K_2 C_e m)^2} \qquad 3\text{–}78$$

$$\tan(l_0) = \frac{-K_2 C_e m}{1 + K_2 C_e n} \qquad 3\text{–}78b$$

Thus the flow through the entry resistance results in a reduced amplitude and a phase lag of the potential fluctuation in the aquifer with respect to that in the open water.
The more general problem of the initiation of the fluctuations in the aquifer when loading effects are significant falls outside the scope of the present analysis. An indication of how this problem may be approached has been given by van der Kamp (1972) through an analysis of tidal fluctuations in a confined aquifer extending under the sea.

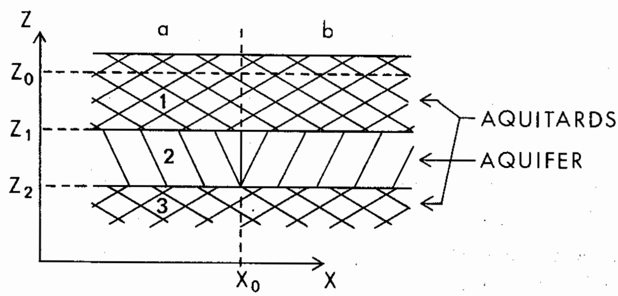

Figure 3–3 Internal vertical boundary

3.4.3. – *Reflection and transmission at an internal vertical boundary*

A change of hydraulic characteristics within an aquifer may lead to important reflection effects. The analysis of such effects given here will be restricted to the case of an abrupt change in the horizontal propagation parameter p across a line parallel to the shoreline as illustrated by figure 3–3.

It is assumed that the boundary between the regions a and b is a vertical plane at $x = x_0$, perpendicular to the x axis. The horizontal propagation parameters p are denoted by p^a and p^b, where the superscripts "a" and "b" denote the regions as in figure 3–3. In general there must be continuity of potential and horizontal flow across the boundary. Now the function $g_j(z)$ which appears in equation 3–71 being part of the special solution for h_j, is determined by the geohydrological properties of the formations and will therefore in general not be the same on both sides of the boundary. It follows that the continuity conditions cannot in general be satisfied by the special solution for h_j in the form of equation 3–71. However, if the vertical potential differences in the aquifer are very small then $g_2(z)$ reduces to a constant, and the continuity conditions can be written:

$$f^a(x = x_0, t) = f^b(x = x_0, t) \qquad 3\text{–}79a$$

$$K^a \frac{\partial}{\partial x} f^a(x,t) \Big|_{x = x_0} = K^b \frac{\partial}{\partial x} f^b(x,t) \Big|_{x = x_0} \qquad 3\text{–}79b$$

With the form of the horizontal potential distributions as given by equation 3–1b, these equations can be written:

$$A_1{}^a e^{p^a x_0} + A_2{}^a e^{-p^a x_0} = A_1{}^b e^{p^b x_0} + A_2{}^b e^{-p^b x_0} \qquad 3\text{–}80a$$

$$K^a p^a (A_1{}^a e^{p^a x_0} - A_2{}^a e^{-p^a x_0}) = K^b p^b (A_1{}^b e^{p^b x_0} - A_2{}^b e^{-p^b x_0}) \qquad 3\text{–}80b$$

It may be assumed for the sake of simplicity that the fluctuations originate in region "a" and that in region "b" far away from the boundary at x_0 they become negligibly small. Then $A_1{}^b = 0$, and solution of equations 3–80 gives for the horizontal potential distributions:

$$f^a(x,t) = A_2^a\, e^{-p^a x} e^{i\omega t} + \\ + A_2{}^a e^{-p^a x_0} \left[\frac{(1 - K^b p^b / K^a p^a)}{(1 + K^b p^b / K^a p^a)}\right] e^{p^a (x - x_0)} e^{i\omega t} \qquad 3\text{–}81a$$

$$f^b(x,t) = A_2{}^a e^{-p^a x_0} \left[\frac{2e^{-p^b (x - x_0)}}{(1 + K^b p^b / K^a p^a)}\right] e^{i\omega t} \qquad 3\text{–}81b$$

The equation for $f^a(x,t)$ consits of two terms, being the incident and the reflected waves respectively. The expression for $f^b(x,t)$ represents the transmitted wave. When the hydraulic conductivity of the aquifer in region "b" is very small so that $K^b p^b$ is much smaller than $K^a p^a$, there is practically total reflection, and the reflected and incident waves have the same amplitude and phase at the boundary $x = x_0$.
The situation to which equations 3–81 apply, namely that of an abrupt transition in conductivity across a line parallel to the shoreline, might seem much too restricted to be of much practical importance. However, observations of tidal fluctuations in fractured sandstone aquifers indicate that such a transition to a less conductive region does often occur. It may be that the repeated back and forth movements of the groundwater in such coastal aquifers have a flushing effect on the fractures, in places resulting in a large increase of conductivity near the shoreline.

3.5 – *Concluding remarks*

The analysis of this and the preceding chapter has been based on what one might call a "classical" approach to the theory of time-dependent groundwater flow, based on Darcy's law and a linear proportionality between changes of water storage and changes of hydraulic potential, as well as other assumptions such as that of homogeneity of the porous materials. The main purpose of the analysis was to work out the consequences of this approach as completely as possible for some simple cases of periodic flow. As far as possible all factors that might influence the flow were considered, with emphasis on the horizontal and vertical flows and the elastic storage in both the aquifers and the aquitards. In this manner the situations for which the various equations apply have been circumscribed in detail. Consequently a comparison of the observational results for such situations with the theoretical predictions can give a good test of how far the "classical" theoretical approach that has been used can account for the flows. To the extent that the theory can be shown to be adaquate, it can then be used with confidence to predict the flows, or to obtain information concerning the hydraulic characteristics of the formations.
Through the analysis presented in this chapter nearly all the equations for special cases of periodic flow, previously derived by other authors, have been integrated in one general approach. In this way the relationship between these and other special cases has been made more clear, and it has been possible to define the ranges of validity of the various special equations in terms of quantitative criteria. Strictly speaking a

particular equation can only be applied to a given situation if all the conditions assumed in its derivation are satisfied.

The equations for the potential distributions and the propagation parameters, derived in this chapter, are written in terms of the angular frequency ω, the thicknesses of the layers D_j, and the hydraulic characteristics K_j, K_j', and S_j'. These variables occur in the equations in various combinations such as the vertical hydraulic resistance $c_j (c_j = D_j K_j')$, the diffusivity S_j'/K_j, or the quantities $S_j c_j$ and $K_j D_j c_j$. The quantities $S_j c_j$ might be referred to as the "time constants" of the layers. When the "time constant" of the overlying aquitard, $S_1 c_1$, is much larger than the characteristic time of the flow ($1/\omega$ for periodic flow) then the flow is confined; when it is smaller the flow is semiconfined or unconfined. In fact the terms "confined" and "unconfined" as applied to time-dependent groundwater flow could be defined in this manner.

In the following chapters empirical results will be presented which serve as an observational check of the theory for periodic flow developed in this and the previous chapter. It will turn out that in many, but not all, cases the theory can give a satisfactory account of the observed potential fluctuations.

4 – DATA ANALYSIS METHODS FOR PERIODIC FLUCTUATIONS

4.1 – *The computation of sinusoidal components*

4.1.1. – *The nature of the data*

The theory developed in the previous chapters refers to periodic variations of hydraulic potential in an aquifer-aquitard system. Such variations can be measured directly, and the resulting data compared to theoretical predictions. In this chapter some problems and methods of analyzing the experimental data will be discussed. Methods of obtaining data on hydraulic potentials such as those utilizing manual taping or water level recorders will not be discussed here.

Once the raw data involving the variations of hydraulic potential with position and time has been obtained, the next problem is to relate these data to the theoretical predictions. This means that an appropriate theoretical model must be selected as discussed in the previous chapters. It also means that the observed potential variations must be split up into the various components due to different kinds of flow, so that each can be considered separately. Such components might for instance be those due to radial flow to a pumped well, or, most important for the present case, the various sinusoidal components due to tidal effects. It is this problem of the splitting up into various components with which this chapter is largely concerned.

4.1.2 – *The general approach for analyzing periodic flows*

The theory for periodic flow developed in the previous chapters is in fact limited to the consideration of flow due to a single sinusoidal fluctuation. This is a legitimate simplification since by the wellknown theorems of Fourier a periodic motion can always be represented as the sum of a series of sinusoidal components of various frequencies. As has been discussed in chapter 2 the equations governing the flow are linear. Each component may therefore be considered as acting independently of the others and may be separately considered. (In principle this linearity is itself a consequence of theoretical assumptions, and should be empirically verified.)

The periodic motions encountered in practice are usually complicated but can be considered as consisting of a small number of significant components. If the theory is to be applied, the frequency, the amplitude and the phase of each component must be determined. Various methods, generally referred to as Fourier analysis, exist for the determination of the components (Dronkers, 1964; Jenkins and Watts, 1969; Godin, 1972 and many others). The choice of a particular method depends on various factors such as whether the frequencies of the components are known in advance, the length and quality of the record, or the practical consideration of the computational facilities that are available. An example of such a Fourier analysis is that carried out by Timmers (1955) for the changes of hydraulic potential due to seasonal changes in the stage of the river Yssel, near Olst, the Netherlands (see chapter 5, section 5.2.3). Since these methods of analysis are wellknown they will not be further discussed here, except for the special case of the analysis of tidal fluctuations in coastal aquifers, which will be discussed in more detail in the next section. A motion is periodic in time if it is repeated exactly at set intervals of time, each equal to the period of the motion. Such pure periodic motions are rarely encountered in practice. Now some form of Fourier analysis can be applied to any variation of potential with time. Whether the theory of periodic fluctuations may be successfully applied to the sinusoidal components thus obtained depends on the extent to which the motion may in fact be considered as periodic in time, and on the extent to which the particular equations that are to be applied are sensitive to non-periodicity. An analysis of the problems involved in deviation from purely periodic motion is not given here. Most of the data that will be presented in the next chapters refer to tidal fluctuations which are indeed periodic to a large extent. For such cases the effects of nonperiodicity are small, and can be further eliminated from the data for phase lag and amplitude by averaging over a time duration of several periods.

4.1.3 – *Computation of tidal components for a one-day water level record*

A common case of periodic groundwater flow is that of the flow in coastal aquifers induced by tidal motions of the sea or of tidal rivers. In fact the empirical results represented in the following chapter are for the most part obtained for this kind of flow. For this reason some methods of analysis for this type of flow will be described in detail in this section. Methods of analyzing and predicting tidal fluctuations in open water are well established (Dronkers, 1964, Godin, 1972). In most cases the periods of the various tidal components are exactly known from astronomical considerations. The tidal fluctuations in groundwater potential will exhibit the same components as occur in the nearby open water, but may also include higher harmonics due to foreshore effects. In principle the methods used to analyze the open water fluctuations can also be applied to the groundwater fluctuations as was done by Carr (1971), who applied harmonic analysis to sixteen-day continous water level records from a row of wells at Borden, P.E.I., Canada, situated within the zone of tidal influence. However various practical considerations usually make it difficult to obtain and analyze the long continuous water level records required for such a harmonic analysis.

For a lenth of time of about one day tidal fluctuations may to a good approximation be considered as consisting of two major components, namely the diurnal component with a period of 1490 minutes, and the semidiurnal

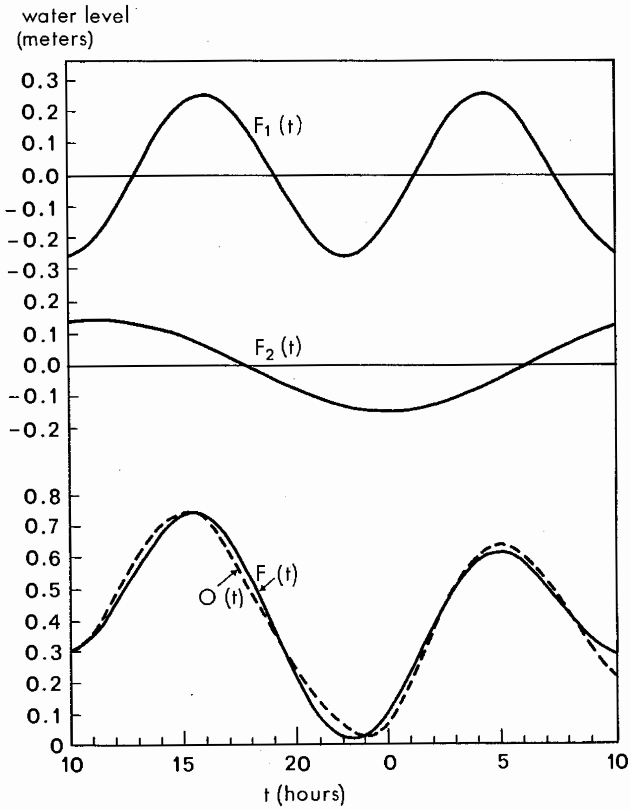

Figure 4–1. Observed water level in well B at Borden, P. E. I., Canada, July 4-5, 1969; and function fitted by one-day least squares fit (t in hours)

Observed water level: $O(t)$

Semidiurnal component: $F_1(t) = 0.258 \cos(0.506t + 0.230)$ meters

Diurnal component: $F_2(t) = 0.151 \cos(0.253t + 1.354)$ meters

Fitted function: $F(t) = 0.415$ meters $+ F_1(t) + F_2(t)$

component with a period of 745 minutes. (In some instances the next higher harmonic with a period of 745/2 = 372.5 minutes may be large enough to be significant.) The diurnal component actually consists of mainly three components with periods of 1436, 1444 and 1549 minutes. Its phase and amplitude will therefore slowly vary from day to day, as the relative phases of these three components vary. Similarly the semidiurnal component comprises mainly three components with periods of 720, 745, and 760 minutes. The approximation of only two components with periods of 745 and 1490 minutes is therefore only valid for lengths of time of not much more than one day. Figure 4–1 illustrates how a tidal motion can be represented as the sum of these two components. The amplitude and phase of these two components can be calculated from a one-day water level record, as described next. Analysis of a longer data record, such as Carr's sixteen-day analysis, allows a resolution of the various components which together form the diurnal and semidiurnal components.

The amplitude and phase of the semidiurnal component only can be determined from the heights and times of the maxima and minima in the water level record. This method, described by Carr and van der Kamp (1969), gives no information on the diurnal component, and can in fact only be used to good effect when the semidiurnal component is larger than the diurnal component. The advantage of this "peak to peak" method of analysis is that it can be carried out quickly, and does not require the use of computer facilities.

For many applications it is desirable to know the amplitude and phase of both components. These can be determined by fairly simple arithmetical methods utilizing the fact that the period of the semidiurnal component is exactly half that of the diurnal component. When computer facilities are available a least squares fit method of analysis affords a simple and reliable means of obtaining the desired information by finding a function of the form:

$$f(t) = M + A_1 \sin(\omega_1 t + d_1) + A_2 \sin(\omega_2 t + d_2) \qquad 4\text{–}1$$

which best fits the observed water level record (ω_1 and ω_2 are the angular frequencies of the semidiurnal and diurnal components respectively). A detailed description of the least squares fit method of analysis is given in numerous texts on applied mathematics or numerical analysis, and is therefore not given here. This method has the advantage that it requires very little time once a computer program is operating, and it is flexible in that it can be easily expanded to consider more components, and can be applied to other types of periodic fluctuations if the frequencies are known.

The data which will be presented in the following chapter have been ana-

Method of analysis	Length of record used	Period of component (minutes)	Well A	Well B	Well C	Well D	Well A	Well B	Well C	Well D
			tidal efficiency				phase lag (radians)			
Harmonic	16 days	745	.519	.321	.190	.047	.363	.869	1.400	1.906
analysis	,,	1490	.576	.410	.279	.087	.333	.730	1.147	1.581
Least squares	1 day	745	.526	.316	.178	.049	.354	.793	1.366	1.889
fit	,,	1490	.584	.424	.344	.099	.380	.789	1.185	1.354
peak to peak	4 days	745	.556	.396	.209	.054	.320	.784	1.408	1.898

Table 4—1. Tidal efficiencies and phase lags for the wells at Borden, P. E. I., Canada, as calculated by three different methods

lyzed mostly by this least squares fit method. Table 4–1 gives data which allow a comparison of the two methods of tidal analysis described above with the sixteen-day harmonic analysis as reported by Carr (1971). For this purpose the records for one typical day of the sixteen were chosen for analysis by the one day least squares method. (These data were made available to the author by Dr. P. A. Carr, whose cooperation is gratefully acknowledged). In the first part of the table the data for the wells A, B, C, and D at Borden, P. E. I., Canada, as determined by Carr from a sixteen-day harmonic analysis are given. The phases and amplitudes of the two components are given relative to those of the tidal fluctuation in the sea, i.e. as phase lag and tidal efficiency (ratio of the amplitude in the well to that in the sea). The phase lags and tidal efficiencies of the diurnal component given here are actually the average of the values given by Carr for the components with 1436 and 1549 minute periods. In the next part of table 4–1 the results of the least squares fit method for one day, and of the "peak to peak" method for four days as reported by Carr, are presented in the same manner.

It can be seen from table 4–1 that the results of both the "peak to peak" and the one-day least squares fit method agree quite well with those obtained by the sixteen-day harmonic analysis.

Figure 4–1 shows the actual water level fluctuations in well B at Borden during the 24 hour period for which the least squares method was applied, together with plots of the two components and the fitted curve in the form of equation 4–1 as determined by the least squares method. This result is typical of the quality of the fit that is usually obtained. It should be mentioned that there are times when a good fit is not possible. This occur when the amplitudes or phases of the two components are changing rapidly, or when the sea level is subject to irregular changes due to storm surges and the like.

In figure 4–2 the results of the least squares fit method for a number of different days are presented graphically. The tidal efficiency and phase lag of the semidiurnal component in two wells are plotted as functions of the amplitudes of the semidiurnal and of the diurnal component in the sea. Well 2-80-0 is about 120 meters from the sea in an unconfined sandstone aquifer. Well 3-207 is located nearby, at about 150 meters from the sea, and open to a fully confined aquifer. Both wells are located in a research site near Cap Pélé, N. B., Canada (see chapter 5, section 5.2, and figure 5–6). These results indicate first of all that at this site the tidal efficiency as obtained by the one-day least squares method is reproducible to within about 10% of the mean, and the phase lag to within about 20% of the mean. Thus the tidal effect can be predicted within this method to within these limits. These results also indicate that the tidal efficiency and phase lag are to a good approximation independent of the amplitude of either component, as might be expected from the linearity of the equations governing the flow.

The one-day least squares fit method cannot reproduce the observed tidal fluctuations exactly since the tidal motions are in fact not strictly periodic with a period of 1490 minutes, and the sea level is often subject to irregular fluctuations due to wind effects and the like.

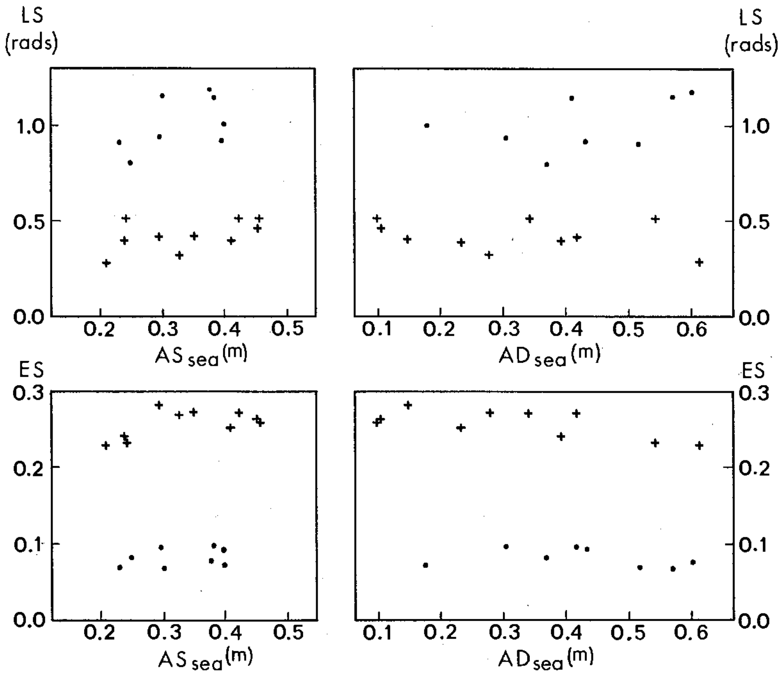

Figure 4–2. Tidal efficiency (ES), and phase lag (LS) for the semidiurnal tidal component, in wells 2-80-0 (+) and 3-207 (.) at Cap Pele, N.B. Canada; plotted versus the amplitudes in the nearby sea of the semidiurnal component (AS_{sea}), and of the diurnal component (AD_{sea})

4.2. – *Elimination of tidal effects from water level records*

The tidal effects in coastal aquifers sometimes make it difficult to observe and analyze other types of potential changes such as those due to pump testing or seasonal changes. Now it has been shown in the previous section that the tidal effects in wells can be predicted because the tidal efficiencies and phase lags are approximately constant on a day-by-day basis. It is therefore possible to eliminate tidal effects from the water level record of a well, usually to the extent that the remaining irregularities are smaller than 10 per cent of the original tidal fluctuations in the water levels of the well. To carry out this calculation the tidal fluctuations of the sea nearby must be measured for about one day and analyzed for the amplitudes and phases of the two components. With the known tidal efficiency and phase lag of each component the tidal potential changes in the well can be calculated, and substraction from the observed potential changes leaves the effects due to other flows.

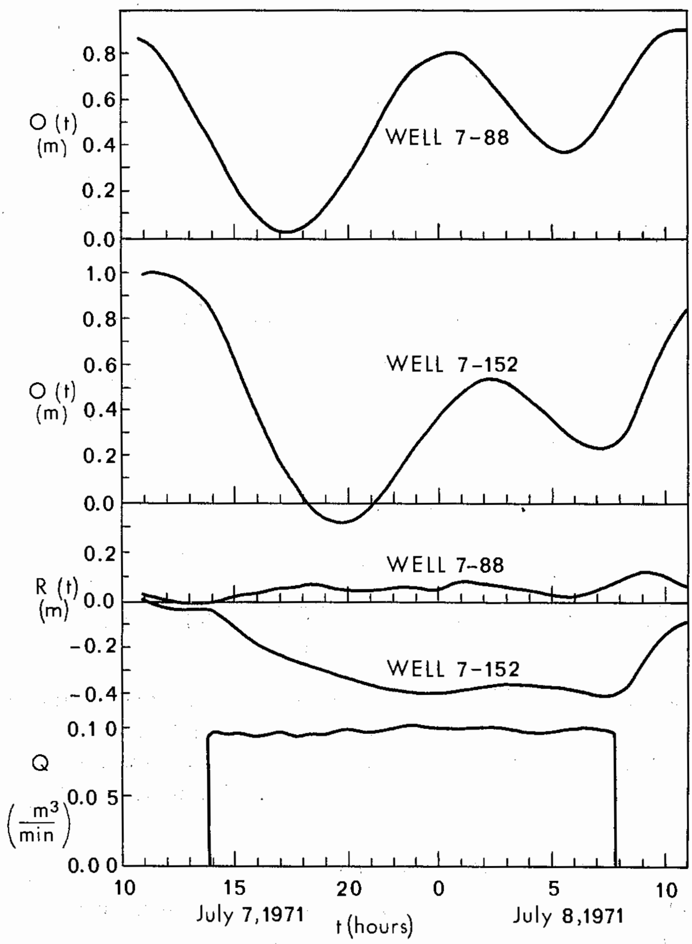

Figure 4—3. Tidal and pumping effects in wells 7-88 and 7-152 at York Pt., P. E. I., Canada. $O(t)$ = observed water level; $R(t)$ = residual water level changes after elimination of tidal effects; Q = pumping rate in a well 196 meters distant, open to the same formation as well 7-152.

A one-day two-component analysis such as illustrated in figure 4–1 can usually approximate the tidal changes closely enough to yield a satisfactory elimination of tidal effects. Figure 4–3 gives an illustration of this method of elimination of tidal effects applied for the drawdowns in wells 7–88 and 7–152 at York Point, P. E. I., Canada. These two wells are positioned one directly above the other, but are open to different aquifers separated by a semipervious layer. Another well 196 meters away and placed in the same formation as well 7–152 was pumped at an approximately constant rate as shown in figure 4–3. The observed water levels in both wells are shown, and also the water level records after elimination of tidal effects. From the corrected water level records it can be seen that there is a clear drawdown in well 7–152, and very little or no drawdown in well 7–88, a conclusion which could certainly not be obtained directly from the observed water level records without correction for tidal effects. These results are fairly typical of the quality of the data that may, with care, be achieved.

The potential changes due to pump testing in coastal aquifers can often not be satisfactorily analyzed without such a correction for tidal effects. Some of the empirical results presented in the next chapter are based on pump testing of the wells used to study tidal effects. For analysis of these pump tests a correction for tidal effects was indispensible.

4.3 – *Well Response*

4.3.1 – *The problem of well response*

The changes of hydraulic potential indicated by an observation device such as a well or piezometer are not necessarily identical with the potential changes within the formation in which it is situated. To the extent that potential changes in a well involve flow of water from the formation to the well or vica versa, there must necessarily be a potential difference between the water in the well and that in the formation at some distance from the well. Thus when the observation device is used to measure changes of potential in the formation the effect of its response may not be negligible and must be considered. In general the importance of this response effect increases with the volume of water per unit potential change that is stored in the device, and with the resistance to flow of the formation and of the lining of the device. It is of course for this reason that pressure transducers, which minimize the flow, are often used to measure potential changes in poorly conducting materials. The following discussion will be restricted to the problem of the response of wells and piezometers.

4.3.2 – *The theory for well response of Cooper et al.*

The response of a well can be determined through observation of the water level changes in the well after an instantaneous withdrawal or addition of a quantity of water. Such a test is often referred to as a "slug test" after Ferris and Knowles (1954). Cooper et al. (1967) have given a detailed analysis of such a test for the case of a well completely penetrating an aquifer. They assume confined horizontal flow in the aquifer, and negligible resistance of the filter lining. Through their analysis they arrive at type curves for the water level in a well after the initial water level change, which allow a determination of the hydraulic conductivity of the formation if the appropiate conditions are satisfied. This method can be useful in that it can give an approximation of the hydraulic conductivity near the various wells in an aquifer.

Cooper et al. did not treat the problem of calculating the potential changes in the formation from the observed water level changes in the well. However, a simple approach to this problem, as described below, may prove satisfactory for many cases.

4.3.3 – *The theory for well response of Hvorslev*

For many observation wells the theory for well response derived by Cooper et al. as described above is not applicable because the effects of vertical flow in the formation or of the resistance of the filter lining are not negligible. For these cases a simple model developed by Hvorslev (1951) might be used. Hvorslev's approach assumes that the rate of flow of water into a well after a sudden withdrawal of water is directly proportional to the difference between the undisturbed potential in the formation (denoted h_f) and the potential of the water in the well (h_w). This proportionality can be written:

$$\frac{\partial h_w}{\partial t} \propto h_f - h_w \qquad \text{4–2}$$

or, as an equation:

$$T_w \frac{\partial h_w}{\partial t} = h_f - h_w \qquad \text{4–3}$$

where the constant T_w is the "time lag constant" of the well, called by Hvorslev the "basic time lag". When h_f is constant, and at time $t = t_0$ the water level in the well is changed by h_0, equation 4–3 gives for the subsequent change of water level in the well:

$$h_w(t > t_0) = h_f + h_0 \exp[-(t-t_0)/T_w] \qquad \text{4–4}$$

This exponential change of water level with time can be used to determine the time lag constant of a well. For many wells and piezometers equation 4—4 seems to give a satisfactory description of the response of the well to a sudden change of water level. Figure 4—4 gives an illustration of two such slug tests for well 1—18 at Cap Pele, N. B., Canada. This well has a filter, open to the formation, of about one meter long and 10 cm in diameter, and is positioned in a sandstone formation. After drilling and cleaning a slug

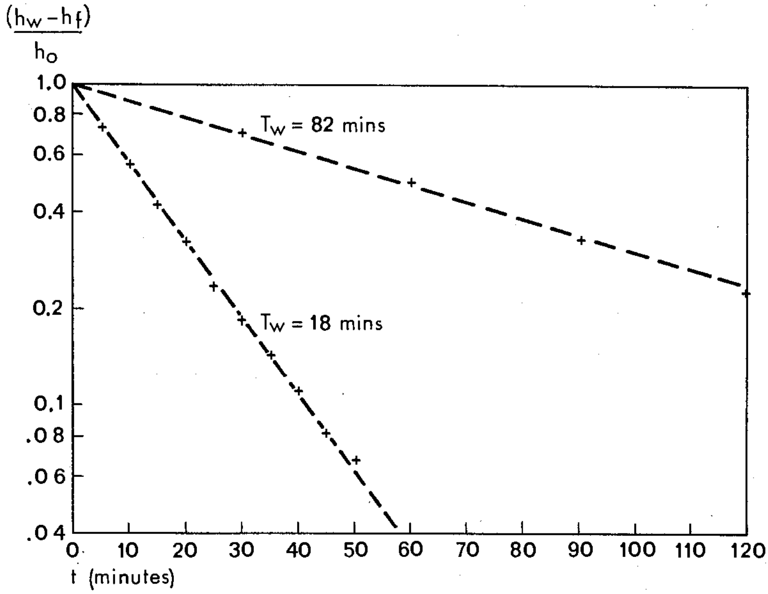

Figure 4—4. Slug tests for well 1-18 at Cap Pele, N. B., Canada. The time lag constant of the well was increased by partially filling the well with sand. Observed water levels are represented by + ; the dashed lines were used to calculate the time lag constants (T_w) according to equation 4—4.

test was carried out on it which yielded a time lag constant of 18 minutes as illustrated in the figure. Some time later the well was partially filled with sand, and another slug test was carried out on it which yielded a time lag constant of 82 minutes. As can be seen from figure 4—4 both tests yielded water level changes corresponding to the exponential behavior described by equation 4—4, at least for that early portion of the text for which reliable measurements of the water level differences could be made.

If the response of a well to a slug test is exponential as described by equation 4—4, the actual potential changes in the formation can be calculated from the observed changes in the well through equation 4—3. In particular, when the water level change in the well is a simple sinusoidal motion expressed as:

$$h_w = A_w \cos(\omega t + d_w) \qquad 4\text{—}5$$

it follows from equation 4—3 that the potential change in the formation at some distance from the well is given by:

$$h_f \doteq A_f \cos(\omega t + d_f) \qquad 4\text{—}6$$

where

$$A_f/A_w = \sqrt{1 + \omega^2 T_w{}^2} \qquad 4\text{—}7a$$

$$d_f = d_w + \arctan(\omega T_w) \qquad 4\text{—}7b$$

Equations 4—7 have been given by Hvorslev (1951)., In view of the over-

time lag constant T_w (mins)	measured				corrected with equations 4—7			
	semidiurnal		diurnal		semidiurnal		diurnal	
	tidal eff.	phase lag (rads)	tidal eff.	phase lag (rads)	tidal eff.	phase lag (rads)	tidal eff.	phase lag (rads)
18	.323	.750	.409	.509	.326	.599	.410	.433
82	.290	.898	.392	.623	.353	.293	.415	.290

Table 4—2. Tidal efficiencies and phase lags in well 1-18, Cap Pele, N. B., Canada, measured for two different time lag constants of the well, and as corrected for well response by equations 4—7.

simplification of the model as expressed by equation 4–3, it is not always quite certain whether they may be applied with confidence. Table 4–2 gives the measured tidal efficiencies and phase lags for the semidiurnal and the diurnal tidal components in well 1–18 at Cap Pele, N. B., for the case when its time lag constant was 18 minutes, and for the case when it was 82 minutes. Together with these are given the tidal efficiencies and phase lags of the potential changes in the formation as calculated through equations 4–7. These should be the same for both time lag constants if equations 4–7 do indeed describe the effect of well response accurately. It would seem from these results that equations 4–7 cannot be relied upon to give a very accurate correction for the response of the well, at least not as regards the phase lags, even when slug tests show exponential behavior as illustrated in figure 4–4.
For the tidal analysis presented in the next chapter equations 4–7 have been used for lack of better, but only where the effect of the inaccuracy of the corrections would be small.

4.4 – *Calculation of the propagation parameters*

It was found in chapter 2 that if the aquifer is thin and for points sufficiently far away from the shoreline the potential distribution for simply periodic flow may be described by an equation of the form:

$$h(x,z,t) = g(z)\,(A_1 e^{px} + A_2 e^{-px})\, e^{i\omega t} \qquad 4\text{–}8$$

where $g(z)$, A_1, A_2, and p are in general complex. The horizontal propagation parameter p can be written:

$$p = n + im \qquad 4\text{–}9$$

where n and m are real and n is defined to be positive. The problem now is to determine p (or n and m) through measurements of the periodic motions in a number of wells. Except for the case of unconfined flow the vertical potential distribution $g(z)$ usually reduces to unity for aquifers. For such cases when the vertical potential differences are very small the positioning of the filters within the aquifer is not critical in the vertical For unconfined flow the effect of the vertical variation of potential can be eliminated if all the observation points are placed at the same height within the aquifer, preferably at the (horizontal) bottom of the aquifer because the vertical potential gradients are smallest there (as follows from equation 3–63). Thus it should be possible to design a row of wells which measures only the horizontal variation of potential (and indirectly the propagation parameter p). Of course it is also possible to install a vertical array of piezometers in order to measure the vertical potential variation.

Equation 4–8 applies for the case when the periodic flow emanates from one or more straight line sources (such as a straight sea shore) so that flow in the y direction (perpendicular to the x and z directions) is negligible. If there is only one such source, so that the fluctuation becomes vanishingly small for large x, and if the thicknesses and the hydraulic characteristics of the formations are constant throughout the region where the periodic fluctuations are significant, then the horizontal potential distribution reduces to the form:

$$f(x,t) = A_2 e^{-px} e^{i\omega t} \qquad 4\text{–}10$$

The real part of this equation can be written:

$$\mathrm{Re}\, f(x,t) = a_0\, e^{-n(x-x_0)} \cos[\omega t - m(x-x_0) + d_0], \qquad 4\text{–}11$$

where a_0 and d_0 are the amplitude and phase constant at the point x_0, usually taken for convenience at the observation point nearest the source of the fluctuation. Thus for this case the amplitude should decrease exponentially with x and the phase lag $m(x-x_0)$ should increase linearly with x. It is therefore possible to check experimentally whether equation 4–10 applies by means of a row of at least three wells. When this equation may be assumed to hold, either from such an experimental check or from knowledge of the hydrogeology of the region, then the quantities n and m can be calculated through equation 4–11 from measurements of the fluctuations in two or more wells. In fact if a_j and a_k are the amplitudes and d_j and d_k the phases lags measured at the ponts x_j and x_k, then:

$$n = \frac{\ln(a_j/a_k)}{x_k - x_j} \qquad 4\text{–}12$$

$$m = \frac{d_j - d_k}{x_k - x_j} \qquad 4\text{–}13$$

In many cases the observed amplitudes and phase lags deviate significantly from the simple form given by equation 4–11. Such discrepancies may be due to gradual changes of the thicknesses or the hydraulic characteristics of the formations in the region where the measurements are carried out, or they may be due to curvature or small extent of the source. When the discrepancies from equation 4–11 are due to such causes the equations developed in chapters 2 and 3 cannot be applied, except perhaps approximately, and further analysis would require a different approach. However, discrepancies from the form expressed by equation 4–11 may also be due to such causes as a multiple shoreline (as of a long narrow island or land tongue) or to reflection effects from a boundary further inland. In such cases the general form of equation 4–8 is applicable:

$$f(x,t) = (A_1 e^{px} + A_2 e^{-px})e^{i\omega t} \qquad 4\text{–}14$$

The problem then is to solve for the complex quantities A_1, A_2 and p. This problem can in principle be solved if measurements from three wells at significant distances apart are available. If the wells are at positions x_j (j = 1, 2, or 3), the measurements will yield:

$$f(x_j,t) = b_j e^{i(\omega t + d_j)} \qquad 4\text{–}15a$$

$$= B_j e^{i\omega t} \qquad 4\text{–}15b$$

b_j and d_j are the amplitude and phase constant measured at the point x_j. The complex quantity B_j can thus be determined through measurement, and equations 4–15 together with equation 4–14 yield:

$$A_1 e^{px_j} + A_2 e^{-px_j} = B_j \qquad 4\text{–}16$$

With the subscript "j" taking the values 1, 2, or 3, equation 4–16 represents a system of three quations in the three unknowns A_1, A_2, and p. The equations are nonlinear and can therefore best be solved through successive approximation methods. When $x_2 - x_1$ and $x_3 - x_2$ are not much larger than $|p|^{-1}$, the following approximate equation for p^2 gives a good first approximation:

$$p^2 = \frac{(x_1 - x_2)(B_2 - B_3) - (x_3 - x_2)(B_2 - B_1)}{(x_1 - x_2)(x_3 - x_2)(x_1 - x_3) / 2} \qquad 4\text{–}17$$

A successive approximation method using this equation has been used for a case described in the next chapter (section 5.4).

5. – DESCRIPTION AND SUMMARY OF THE EMPIRICAL DATA

5. 1 *Introduction*

5.1.1. *The subject of this chapter*

In chapters 2 and 3 a theory has been developed for the propagation of periodic fluctuations through a three-layer system of aquitards and thin aquifers. The question is now to what extent the theory can account for the flows encountered in practice, and how it can best be applied in the pratical problem situations encountered by the geohydrologist. In an attempt to give at least a partial answer to this question empirical results from a number of different types of periodic flow will be presentend and analyzed in the light of the theory, in this and the following chapter.
In this chapter the empirical data from a number of different sites will be described and summarized. The analysis of these data in its relation to the theory is left for the following chapter. In a sense therefore the material presented in this chapter is included for the sake of completeness. It is not essential to an understanding of the following chapter, but it provides the background information for the crucial comparison of theory and experiment which is treated there.

5.1.2. *The hydrostratigraphic model*

The hydrostratigraphic model for which the theory has been developed has been given in chapters 2 and 3, but it is outlined again here, since the data will be described in terms of this model. Figure 5–1 illustrates the model.

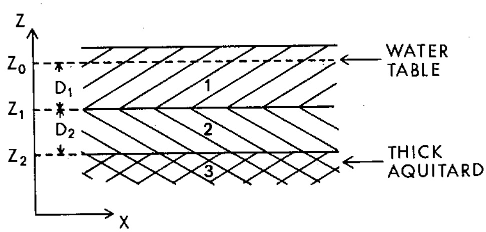

Figure 5–1 The hydrostratigraphic model

The layers 1, 2, and 3 have thicknesses D_j, horizontal hydraulic conductivities K_j, vertical hydraulic conductivities K_j', and specific storage coefficients S_j'. The storage coefficient at the water table ($z=z_0$) is S_0. For all the empirical cases treated in this chapter layer 1 is either an aquitard or absent altogether. It should be noted that when layer 1 is thick aquitard, isolating the flow in the aquifer (layer 2) from the flow above, and vice versa, then the theory remains applicable for the flow in layer 2 even when layer 1 is not bounded above by the water table, as illustrated in figure 5–1, but by other saturated formations. The numbering system for the formations used in figure 5–1 will be used in the description of all the empirical cases, i.e. in each case the main aquifer will be referred to as layer 2, and the aquitard above it as layer 1.
The following quantities are also frequently used:

$$S_j = S_j' D_j \qquad 5\text{–}1$$

which is the storage coefficient of layer j, and

$$c_j = D_j / K_j' \qquad 5\text{–}2$$

which is the vertical hydraulic resistance of layer j.

5.1.3. *The type of data required*

In essence two types of data will be sought for each site that is studied. One of these is data on the horizontal and vertical propagation of periodic fluctuations. The other is information on the geohydrological properties of the formation obtained by tests other than those using periodic flow.
Data on the propagation of the periodic fluctuations consist of values of the amplitudes and phase constants at various points in the formations. For most cases the horizontal propagation is described by means of the propagation parameters n and m, which are calculated on the assumption that the horizontal potential distribution can be described by a function of the form (equation 2–17):

$$f(x,t) = (A_1 e^{px} + A_2 e^{-px})\, e^{i\omega t} \qquad 5\text{–}3$$

where:

$$p = n + im \qquad 5\text{–}4$$

The quantities n and m are calculated by the methods described in chapter 4 (section 4–4).
Information on the geohydrological properties of the formations (D_j. K_j, K_j', S_j', and S_0) is obtained or estimated from various sources such as drilling logs, core tests, grain-size analysis, pump tests, etc. Except for the case of unconfined flow vertical potential gradients within the aquifers are nearly always small, with the results that little information is available on K_2'. Similarly the horizontal flow in the aquitards is usually negligible, and virtually no information is available on K_1.

5.1.4. *Sources of the data*

For two cases, namely those of the two aquifers at Cap Pele, N. B., Canada, the bulk of the measurement work was carried out by the author, or under his supervision. The other data that are quoted are abstracted from the results of work done by others, both published and unpublished. For all these the sources will be described or acknowledged. For several cases the data given by others have been reanalyzed to put it into a form more useful for the present purposes.
Most of the empirical data are summarized in table 5–2 at the end of this chapter.

5.2. *Data from the Tielerwaard (Dalem and Hellouw), The Netherlands*

A detailed study of the hydrogeology of the western part of the polder Tielerwaard, along the river Waal in the Netherlands has been published in a report entitled "De Waterbehoefte van de Tielerwaard-West" (The Water Requirement of the Tielerwaard-west), (ed. de Ridder, Blok, and Colenbrander, 1961). These studies include data on the propagation of tidal fluctuations through the aquifer. and data on pump tests and other aquifer tests. The variety and quality of the data warrant further analysis.
A simplified sketch of the hydrostratigraphy near Dalem in the Tielerwaard is given in figure 5–2. It is based on the geohydrologic profiles given in the Tielerwaard report by Verbraeck and de Ridder (1961). The main geohydrologic units are not as homogeneous as indicated by figure 5–2, but have been taken as homogeneous for the present purpose since the theory that is to be applied assumes homogeneity in any case. A more detailed description of the stratigraphy has been given by Verbraeck and de Ridder, and by Kruseman and de Ridder (1970, p. 71). The covering aquitard has a thickness of about 8 meters, and the aquifer has a thickness of about 36 meters.
A pump test was carried out at the location shown in figure 5–2. The draw-

down data of this pump test have been analyzed in a variety of ways by Kruseman and de Ridder (1970). They obtain a value for $K_2 D_2$ of 2000 m^2/day, and for c_1 of 450 days. Since their analyses are based on the assumption of semiconfined flow with a fixed water table, the value of the elastic storage coefficient which they find (2.0×10^{-3}) includes the effect of storage in the overlying aquitard. It is therefore not equal to S_2 but to $S_2 + S_1/3$, as indicated in chapter 3 (equation 3–65), and by Hantush (1960).

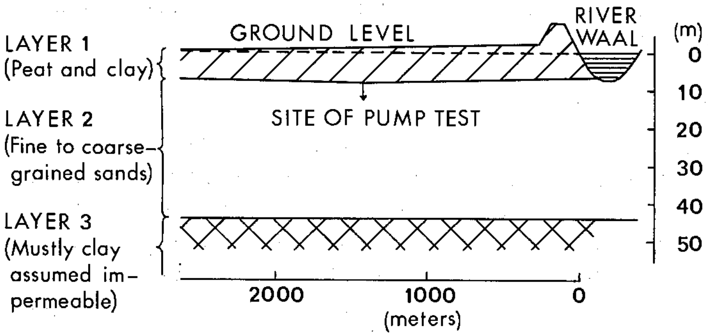

Figure 5–2 The hydrostratigraphic profile in the Tielerwaard near Dalem, The Netherlands (after Verbraeck and de Ridder, 1961)

The early time drawdown data for this pump test can be analyzed to give S_2 only, by the method of Cooper and Jacob (1946) which involves a plot of drawdown versus the logarithm of time since start of pumping. This method is also described by Kruseman and de Ridder (1970). As indicated by the results of Hantush (1960) the method may be applied with a fair degree of accuracy if $(r/4D_2)\sqrt{(K_1' S_1'/K_2 S_2)}$ is less than 0.01; if $u = r^2 S_2/4K_2 D_2 t$ has values in the interval between 0.1 and 0.01; and if t is less than $c_1 S_1/10$ (r is the distance from the pumped well to the observation well, t is the time since start of pumping). Analysis with the method of Cooper and Jacob of the drawdown data given by Kruseman and de Ridder gives average values of $K_2 D_2 = 2020$ m^2/day (in good agreement with the value from "semiconfined analysis"), and $S_2 = 1.2 \times 10^{-3}$.
With $S_2 + S_1/3 = 2.0 \times 10^{-3}$ as given by Kruseman and de Ridder, a value for S_1 is obtained of 2.4×10^{-3}. With these results checking back shows that the conditions mentioned above for the applicability of the Cooper-Jacob method are fairly well satisfied.
Since the water table in the polder is kept well below that of the river a measurement of the stationary hydraulic potentials in the aquifer near the river can yield a value for the product $K_2 D_2 c_1$ by the method of Mazure

(1932) (see chapter 3, equation 3–70). Colenbrander (1961) reports an analysis of this sort in the Tielerwaard report. He finds that along the Dalem profile (figure 5–2) the value of $K_2 D_2 c_1$ is about 2.0×10^6 m^2. If the value of $K_2 D_2$ indicated by the pump test (2000 m^2/day) also holds near the river, then near the river the value of c_1 must be about 1000 days.
The geohydrological properties of the formations as found by the methods described above are summarized in table 5–2.
Wesseling and Colenbrander (1961, Tielerwaard report) have given data on the propagation of the semidiurnal tidal component (frequency 12.14 rads/day) along the line of figure 5–2. Their results for the propagation parameters are given in table 5–2.
Besides being subject to tidal fluctuations the river Waal is also subject to other fluctuations in water level due to various seasonal effects, of longer duration than the tidal motions. These fluctuations also show up in nearby wells. Colenbrander (1961, Tielerwaard report) reports such a set of fluctuations along the Dalem profile. His data are reproduced in figure 5–3a. A simple sinusoidal function with a period of 28 days (frequency 0.225 rads/day) has been fitted to these curves with the least squares fit method described in chapter 4 (section 4.1). The results for the propagation parameters between wells 2 and 4 are given in table 5–2. Inspection of the water level data over a longer length of time shows that the motion is certainly not periodic with a period of 28 days. Nevertheless the results obtained through the assumption of a 28 day periodicity are fairly reliable because the phase lags are very small as compared with the period of the wave so that the effect of nonperiodicity (or initial conditions) is small. The data on this long-period fluctuations have been included in spite of their questionable reliability because, compared with the tidal motions, they yield an interesting illustration of two flow types in the same aquifer.
Wesseling and Colenbrander (1961, Tielerwaard report) also give data on the propagation of tidal motions near Hellouw, also in the Tielerwaard. At this point D_1 = 6 m and D_2 = 44 m. Grain size analysis gives $K_2 D_2$ = 2500 m^2/day. Measurement of the stationary water levels gave $K_2 D_2 c_1 = 2.0 \times 10^6$ m^2, resulting in a value for c_1 of about 800 days. If the specific storage coefficients are assumed to be the same as at Dalem, the values of S_1 and S_2 are 1.8×10^{-3} and 1.5×10^{-3} respectively. These data for the Hellouw site (summarized in table 5–2) have been given in the Tielerwaard report by the various authors mentioned above with respect to the Dalem site.
An illustration of the observed water level fluctuations at Hellouw, given by Wesseling and Colenbrander (1961) is reproduced in figure 5–3b. By the least squares fit method these data have been analyzed for the amplitudes and phase lags of both the semidiurnal tidal component (frequency 12.14

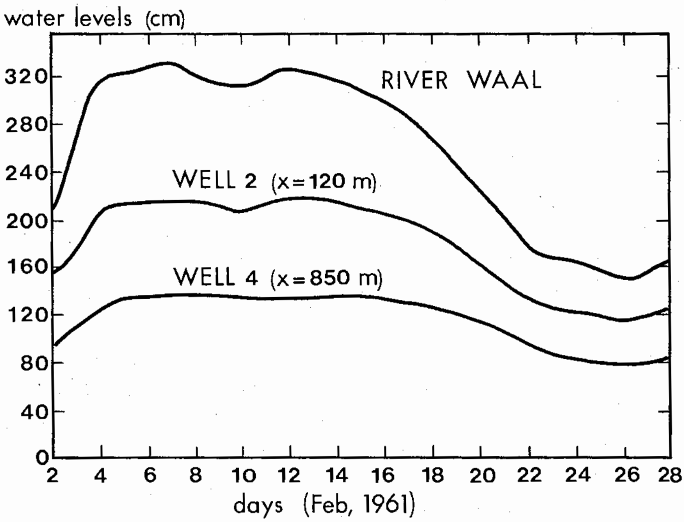

Figure 5–3a Water level fluctuations in wells near Dalem in the Tielerwaard (after Colenbrander, 1961)

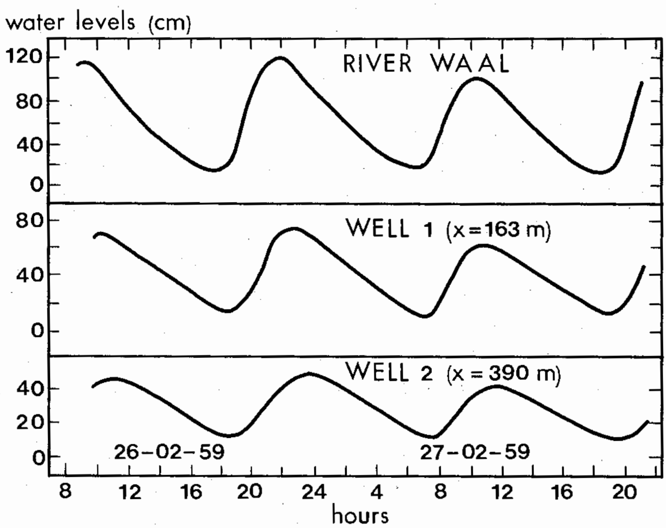

Figure 5–3b Tidal water level fluctuations in the Tielerwaard near Hellouw (after Wesseling and Colenbrander, 1961)

rads/day) and of the next higher harmonic (frequency 24.28 rads/day). The results for the propagation parameters, calculated for the data from wells 1 and 2, are given in table 5–2.

5.3. – *Data from the dune-water catchment area of Amsterdam, near Zandvoort, The Netherlands*

In connection with the development of water supplies for the city of Amsterdam, extensive geohydrological investigations have been carried out in the dune-water catchment area south of Zandvoort, the Netherlands, by the agency of the Municipal Waterworks of Amsterdam.

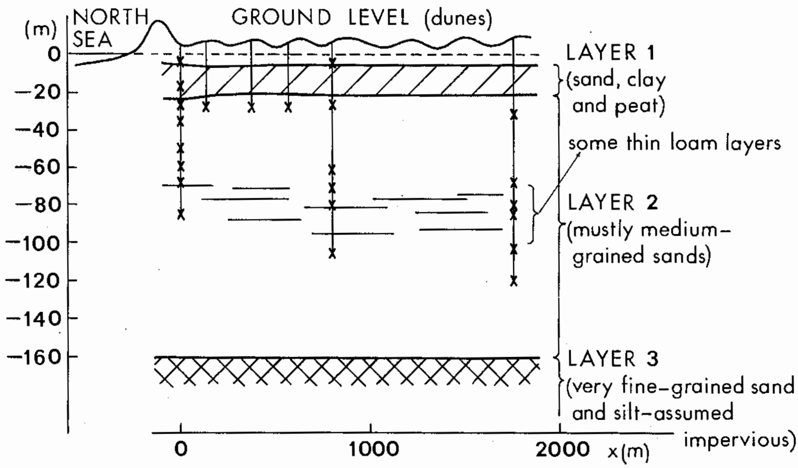

Figure 5–4 Hydrostratigraphy and position of the piezometers along row W in the dune water catchment area of Amsterdam near Zandvoort, The Netherlands. The positions of the piezometers are marked by x,

The data on tidal fluctuations which will be presented here, have been collected from a row of piezometers (row W, piezometer sites W1800 to W 0), the positions of which are shown in figure 5–4, along with the main hydrostratigraphic units.
The geohydrology of this region has been described by Huisman (1957). The aquitard (layer 1) has been estimated by Huisman to have a vertical hydraulic resistance of about 750–2000 days in this part of the region, and the value of $K_2 D_2$ is about 4500 m^2/day. The thin loam layers between –70 m and –100 m have been estimated by Huisman to have a vertical hydraulic

resistance between 0 and 200 days, generally increasing with distance from the sea. The base (layer 3) is composed of very fine sand and silt and is assumed impermeable.

A pump test has been carried out at a site about 4 km inland from the sea, and roughly in line with the row of piezometers shown in figure 5–4. (Data of this pump test are unpublished and were made available to the author through the cooperation of A. J. Roebert and R. A. Schuurmans of the Municipal waterworks of Amsterdam). The pumped filter was open only to a part of the upper portion of the aquifer above the loam layers lying between about –20 and –80 m. Analysis was performed by the method of Cooper and Jacob (1946) as described above with respect to the pump test at Dalem in the Tielerwaard. The analysis yields a value for the storage coefficient of this part of the aquifer of 7.0×10^{-4}. If the specific storage of the deeper part of the aquifer (–80 to –160 m) is the same, the total storage coefficient S_2 equals $(140/60) \times 7.0 \times 10^{-4} = 16.3 \times 10^{-4}$. An accurate estimate of the storage coefficient of the aquitard S_1 could not be

x = 0 m			x = 806 m			x = 1760 m		
Depth below sea level (m)	A_{rel}	Phase lag (rads.)	Depth below sea level (m)	A_{rel}	Phase lag (rads)	Depth below sea level (m)	A_{rel}	Phase lag (rads.)
3	<0.01	–	3	<0.01	–			
16.5	0.94	–0.008	25.4	0.27	0.71	31	0.083	1.18
25.5	1.00	.000	60.4	0.27	0.70	67	0.067	1.38
36.5	1.00	–.004	70.4	0.28	0.71	79.5	0.067	1.47
48.5	1.02	–.014	79.4	0.26	0.72	84	0.070	1.42
58.5	1.00	.001	105.4	0.27	0.60	102.5	0.042	1.75
66.5	0.99	–.005				120	0.067	2.10
84.5	0.99	–.016						

Table 5–1 Variation with depth of the relative amplitudes and phase lags of the semidiurnal component in the deep aquifer of the dune-water catchment area of Amsterdam (row W), near Zandvoort, The Netherlands. The relative amplitudes (A_{rel}) are the ratio of the amplitude with respect to the amplitude in the piezometer at x=0, depth = 25.5 m. The phase lags are the difference in phase with respect to the phase in the piezometer at x = 0 m, depth = 25.5 m.

made on the basis of the pump test data, although analysis by semiconfined flow methods does indicate the effect of storage in the aquitards. In table 5–2 the values for the geohydrological properties of the formations that have been finally been adopted are given.

Tidal fluctuations in the piezometers of row W shown in figure 5–4 were measured during three separate 24 hour periods in February and March 1948. (The original water level data were made available to the author through the cooperation of A. J. Roebert of the Municipal Waterworks of Amsterdam.) The tides in the North Sea are mainly semidiurnal but at the times of measurement the diurnal component was of large enough amplitude to be separately considered. The amplitudes and phase constants of both components were calculated by the one-day least squares fit method described in chapter 4, section 4.1, relative to those in the piezometer at –25.5 m in the piezometer site nearest the sea. The results were averaged

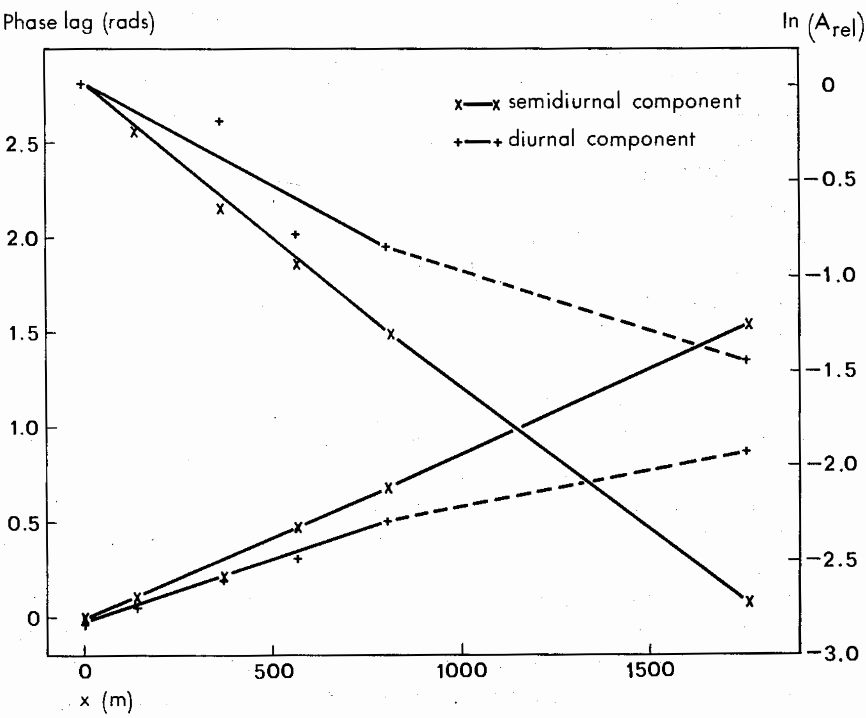

Figure 5 5 Phase lags and relative amplitudes (A_{rel}) along row W near Zandvoort. Data are quoted relative to the piezometer at x = 0, and at depth 25.5 m below sea level.

over the dree days of measurement. The large number of piezometers gives a detailed picture of both the horizontal and vertical potential distributions within the aquifer.
In table 5–1 the relative amplitudes and phases of the semidiurnal component are listed for the three points where a number of piezometers are placed at various depths. These figures thus relate to the vertical potential distribution. The data for the piezometers at x = 1760 m are less accurate than the others because they represent only two days of measurement, instead of three, and because the amplitudes were very small at this point (about 2 cm). It may be noted that the piezometers above the confining aquitard showed no measurable tidal fluctuations. The data given in table 5–2 indicate that at least within the upper part of the aquifer above the loam layers variations of potential with height are practically negligible.
The horizontal potential distribution is presented graphically in figure 5–5 in terms of the phase lag and the natural logarithm of the relative amplitude plotted versus the distance along a line perpendicular to the coast line. The data for x = 0, 806 and 1760 m are taken as most accurate because these represent the average of results from a number of piezometers at various depths. The data for the diurnal component at x = 1760 m are not reliable because of its small amplitude at this point (about 3 cm), and because of the disturbing influence of other effects with an approximately daily cycle, such as evapotranspiration or daily pumping cycles. In an other row of piezometers (row Z), about 4 km to the south of row W, virtually identical results were obtained for the semidiurnal component. The propagation parameters, calculated from the data for the piezometers at x = 0 and x = 806 meters, are given in table 5–2.

5.4. – *Data for a water-table aquifer at Cap Pele, N. B., Canada*

Extensive geohydrological investigations have been carried out at a site on the Northumberland Strait shoreline near Cap Pele, N. B., Canada. This research on groundwater flow in coastal aquifers was carried out under the auspices of the Hydrology Research Division, Department of the Environment, Canada.
The main hydrostratigraphic formations and the position of wells and piezometers at this site are shown in figure 5–6. For the moment only the upper sandstone unit will be considered and will be denoted as layer 2, in accordance with the numbering system of the hydrostratigraphic model. Only the wells and piezometers used for tidal measurements are shown in figure 5–6. For each well filter the horizontal hydraulic conductivity was determined by means of slug tests analyzed with the method of Cooper et al (1967) as

discussed in chapter 4 (section 4.3). These values are shown in figure 5–6 beside the filters. These results indicate that in the upper sandstone unit there is a change of horizontal hydraulic conductivity by about a factor 4 or 5 at about 200 m from the sea. This conclusion is born out by the change in the slope of the water table as shown in figure 5–6. The change in conductivity leads to reflection of the tidal fluctuations moving inland through the aquifer, as will be shown below.

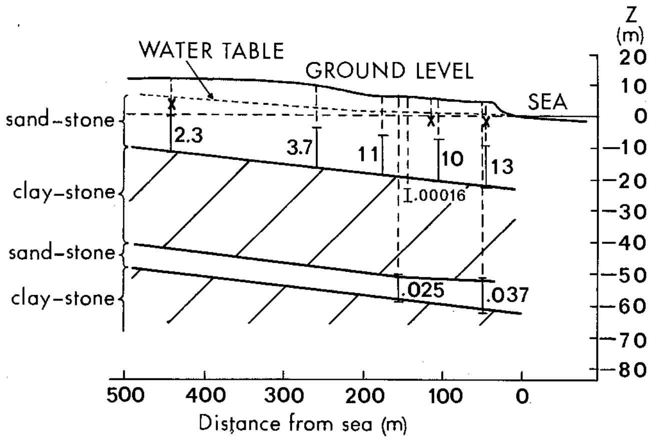

Figure 5–6 Hydrostratigraphic profile and position of the wells and piezometers at Cap Pele, N. B., Canada. The well filters are marked by a solid vertical line, the positions of the piezometers are marked by X. Beside each well filter is given the horizontal hydraulic conductivity in meters/day as determined by means of slug tests.

In the part of the upper aquifer near the sea a pump test was carried out. Tidal effects were eliminated from the observed water level records in order to obtain the drawdowns due to pumping, by the method described in chapter 4 (section 4.2). Observation wells placed in various directions with respect to the pumped well indicated no significant horizontal anisotropy. The drawdowns are of a "delayed yield" type as described by Boulton (1954, 1963). An analysis of the drawdowns was carried out by Boulton's method, with a reinterpretation of the results, as explained in chapter 3 in connection with the effect of vertical flow in the aquifer (see the discussion following equation 3–39).

The results of the pump test analysis for a 1200 minute duration of pumping are summarized in table 5–2. The average horizontal hydraulic conductivity calculated from the pump test results is 13.3 m/day, in fair agreement with the slug test results of about 11 m/day for the near-sea part of the aquifer.

Analysis of the drawdowns by the method of Jacob (1940) using a plot of drawdown versus the square of the distance from the pumped well indicated that the value of S_0, the storage coefficient at the water table, increases with the duration of the test even after the effects of vertical flow have become negligible. This result is not altogether surprising since the value obtained

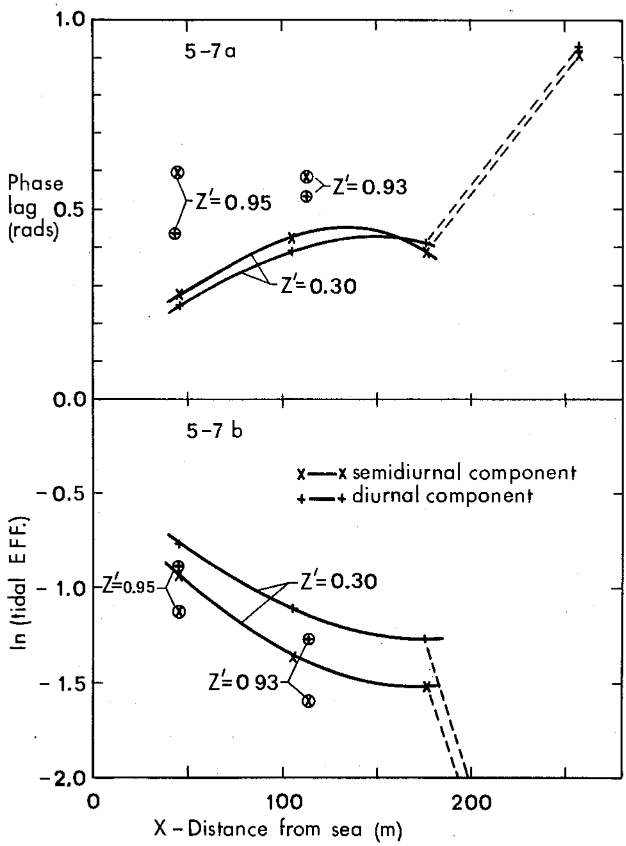

Figure 5–7a and 5–7b Tidal efficiencies and phase lags in the water table aquifer at Cap Pele, N. B., Canada. Data from piezometers near the water table ate circled. Z′ is the relative height of the observation point (height of midpoint of filter or piezometer above base of aquifer divided by thickness of aquifer).

for a 1200 minute test (S_0 = 0.015) is far smaller that the porosity of the sandstone (0.22). In other words it seems that the value of S_0 depends on the duration of the flow.

Measurements of tidal fluctuations were carried out in all the wells and piezometers shown in figure 5–6. The tidal efficiencies (ratio of amplitude in the well with respect to that in the sea) and the phase lags with respect to the tides in the sea of both the diurnal and the semidiurnal tidal components were determined as an average from a number of one day observation periods. The results for the water table aquifer are shown in figure 5–7a and 5–7b. Since vertical variations of potential are important in this case, the horizontal potential distribution is determined only by means of the wells open to the bottom half of the aquifer. The reflection effects due to the change in hydraulic conductivity at about 200 m from the sea show up clearly. Between 176m and 256 m distance, for instance, the tidal efficiency of the semidiurnal component drops from 0.222 to 0.017. The horizontal potential distribution therefore must be considered as the sum of two waves: one moving in from the sea, and one reflected back from the semipermeable vertical boundary. The amplitudes and phases of the two waves were determined with the method described in chapter 4 (equations 4–14 to 4–17) which also yields values for the propagation parameters n and m (given in table 5–2). The tidal efficiencies and phase lags calculated from a superposition of the two waves are shown by the solid lines in figures 5–7a and 5–7b.

Also shown in figures 5–7 are the tidal efficiencies and phase lags measured in two piezometers placed just beneath the water table. These data have been corrected for the response of the piezometers by the method of Hvorslev (1951) described in chapter 4 (equations 4–7). The time lag constants of these piezometers (14 and 18 minutes) are small enough with respect to the phase lags that the error introduced by the correction for well response will be small. The data show that near the water table the tidal efficiency is less than at greater depths in the aquifer, and the phase lag is larger. These measurements of the vertical variation of potential for unconfined flow will be further analyzed in the next chapter.

5.5. – *Data from several other sites*

5.5.1 – *Cap Pele, N. B., Canada – deep aquifer.*

Measurements of tidal fluctuations were carried out in the deep aquifer at Cap Pele, N. B., Canada, shown in figure 5–6. The propagation parameters, calculated from the difference of tidal efficiency and phase lag between the two observation wells, are given in table 5–2.

A pump test was also carried out on the wells used for tidal measurements. The observed drawdowns, after elimination of tidal effects, were analyzed by the method of Papadopoulos and Cooper (1967) for the case when storage in the wells is significant. The results for K_2D_2 and S_2, based on the assumption that flow from the aquitards is negligible, are given in table 5–2. The value for K_2 (0.025 m/day) obtained in this way agrees well with the value for K_2 from slug tests (0.031 m/day). Slug tests on a well in the same aquifer about 1000 m further inland gave a value for K_2 of 0.06 m/day, indicating that there is probably no drop in conductivity with distance away from the sea.

Tests on a drill core gave a vertical hydraulic conductivity for the claystone aquitard of about 2×10^{-6} m/day. Allowing a factor 15 for the extra conductivity due to fractures yields an estimate for c_1 of 10^5 days. The value of S_2' is very small, and therefore the value of S_1' is assumed to be correspondingly small.

5.5.2. *Borden, P. E. I., Canada*

Carr (1971) carried out extensive measurements of tidal fluctuations in a row of wells near Borden, P. E. I., Canada. He determined the tidal efficiencies and phase lags of the two major tidal components using harmonic analysis of sixteen continuous days of water level records. These results were mentioned in chapter 4 with respect to the description of various methods of computing tidal components (table 4–1). Both the tidal data and slug tests indicate that at Borden, as in the water-table aquifer at Cap Pele, there is a drop in horizontal hydraulic conductivity at some distance from the sea (between 120 and 180 meters for the Borden aquifer). The propagation parameters calculated from Carr's data for the two wells nearest the sea are given in table 5–2.

The geohydrological profile at the site has been given by Carr, and indicates a layered sandstone aquifer, and a confining claystone layer of about 5 meters thick. The geohydrological properties of the formations, as estimated from the information given by Carr, and from a pump test carried out about 2 km away, are summarized in table 5–2.

5.5.3. *Olst, The Netherlands*

Timmers (1955) measured the effect in a row of wells of seasonal fluctuations in stage of the river Yssel near Olst, the Netherlands. He gives graphically the amplitudes and phases of the two slowest components (periods of 28 and 14 weeks) as determined by harmonic analysis. In order to determine the propagation parameters these data have been reana-

lyzed by fitting the best straight line through the data points. The results for the propagation parameters are given in table 5–2.
The stratigraphy at the site is also discussed by Timmers. Basically it consists of a sand aquifer about 15 meters thick, overlain by a clay layer with a thickness varying between 0 and 5 meters, and underlain by an impervious loam layer. The hydraulic properties of the formations as estimated from this information are given in table 5–2.
Timmers gives a good and thorough analysis of the results. They are quoted here because they are an example of long-term periodic fluctuations and because they illustrate the case of unconfined flow, as will be shown in the next chapter.

5.5.4. *Prunjepolder, The Netherlands*

Wesseling (1960) gives the results of measurements of tidal fluctuations in several rows of piezometers in the Prunjepolder, situated in the south-west Netherlands. His results for the average propagation parameters in rows 2 and 5 are given in table 5–2.
The stratigraphy at this site has been described by Wesseling (1960), and by van Dam and de Ridder (1960). It is similar to that of the Tielerwaard as illustrated in figure 5–2, except that there are some thin semipermeable layers within the aquifer itself, and the base is probably not entirely impervious.
Van Dam and de Ridder (1960) indicate that the value of $K_2 D_2$ as estimated from grain size analyses is about 400 m^2/day.
For row 2 the thickness of the covering aquitard is about 8 m, and for row 5 it is about 1 m. This difference in the thickness of the covering layers allows observation of an interesting contrast of the results for the propagation of tidal fluctuations, and is in fact the main reason for the inclusion of these data. Grain size analysis at one point in row 2 as quoted by Wesseling gave a value for the vertical hydraulic resistance c_1 of 750 days. The value of c_1 in row 5 is estimated to be about 100 days, on the assumption that the value of c_1 is proportional to the thickness of the aquitard. Analysis of the stationary potentials in the piezometers of row 5 by the method of Mazure (1932), (equation 3–70) indicates that the product $K_2 D_2 c_1$ equals about 40,000 m^2, in good agreement with the above estimates. The data are summarized in table 5–2.

5.5.5. *Oude Korendijk, The Netherlands*

De Ridder and Wit (1965) reported the results of geohydrological investigations in the polder Oude Korendijk, in the south-western Netherlands, involving observations of tidal fluctuations, pump tests, and other methods of determining the hydraulic characteristics of the formations. Their results for the propagation parameters of the semidiurnal tidal component in the upper aquifer of row 1 is given in table 5–2.

The geohydrologic profile along row 1 has been given by de Ridder and Wit. It is similar to the Tielerwaard profile (figure 5–2), except that the base of the upper aquifer is a layer of clay and fine sand only about 6 to 12 meters thick beneath which is another aquifer. For the analysis of the tidal data it will be assumed that flow in this base is negligible. A pump test and other tests, as quoted by de Ridder and Wit, give a value for $K_2 D_2$ of the upper aquifer of 300 m^2/day. The value of c_1 is estimated by de Ridder and Wit to be about 1600 days on the basis of laboratory tests on undisturbed samples. Another pump test in the vicinity, with the same thickness of the upper aquifer, has been analyzed by Kruseman and de Ridder (1970, p. 69) who arrive at a value for S_2 of 2×10^{-4}. The geohydrological characteristics of the formations are summarized in table 5–2.

5.5.6. *Summary of the empirical results*

Empirical results from the sites described above are summarized in table 5–2. Several remarks must be made about the data presented in the table. Estimates of the possible error are indicated by means of asterisks and brackets. No asterisk or bracket means that the figure quoted is probably accurate to within 10%. One and two asterisks indicate possible errors of 30% and 50% respectively. Brackets around a figure indicate that it is no more than an estimate, probably accurate to within an order of magnitude, and possibly fairly reliable. These estimates of the possible errors are based on a rather subjective evaluation of a variety of factors, but should give a good indication of the reliability of the data.

The value of S_0 is unknown for all cases except that of the water table aquifer at Cap Pele. Prickett (1965) quotes data for S_0 from a number of different pump tests, varying from 0.005 for a 3-hour test to 0.25 for longer tests. In view of the variation of S_0 with the duration of the flow, S_0 has been estimated as 0.01 for tidal fluctuations, and 0.1 for the fluctuations with longer periods at Dalem and Olst.

The value of the specific storage coefficients of the aquitards, S_1', where not known, has been estimated at 2×10^{-4} m^{-1}, and the value of S_1 calculated accordingly. Similarly the value for the specific storage coefficient of the aquifers, S_2', where not known, has been estimated at 2×10^{-5} m^{-1}.

The vertical hydraulic conductivity of the aquitards, K_1', where not known, has been estimated at 0.01 m/day, and the value of $c_1 = D_1/K_1'$ calculated accordingly. The value of K_2', where not known, has been estimated at

0.1 x K_2 (horizontal-vertical anisotropy of 10) and c_2 calculated accordingly.

No data are quoted for the underlying aquitard, layer 3. In all cases very little data on this layer are available, and it is assumed either to be imper-

SITE	Geohydrological characteristics of the formations from drilling and tests other than those utilizing periodic fluctuations								Propagation parameters for periodic fluctuations		
	D_1 (m)	D_2 (m)	K_2D_2 (m^2/day)	S_0	S_1 (10^{-4})	S_2 (10^{-4})	c_1 (days)	c_2 (days)	freq. rads day	n (m^{-1} x 10^{-3})	m (m^{-1} x 10^{-3})
Zandvoort	17	140	4500	(.01)	(34)	16*	1000**	$<$200	12.14	1.62	0.86
,,									6.07	1.07	0.67
Tielerwaard (Hellouw)	6	44	2500	(.01)	18*	15*	800	(8)	24.28	2.7	1.9
,,									12.14	2.0	1.6
Tielerwaard (Dalem)	8	36	2000	(.01)	24*	12*	1000	(7)	12.14	2.37	1.70
,,				(0.10)					0.225	0.82	0.16*
Cap Pele (water table)	0	21	280	.015*	0	2*	0	75*	12.14	10.4	3.0
,,									6.07	8.7	2.7
Cap Pele (deep)	30	8	0.20*	.015*	(0.6)	.03**	(10^5)	(1800)	12.14	12.1	4.76
,,									6.07	9.01	3.64
Borden	5*	15*	4**	(.01)	(10)	1.3**	(500)	(55)	12.14	7.72	8.37
,,									6.07	5.50	6.57
Olst	2**	15*	750**	(.10)	(4)	(3)	(200)	(3)	0.028	2.2*	0.64*
,,									0.014	1.6	1.02
Prunjepolder (2)	8*	40*	400*	(.01)	(16)	(8)	750*	(40)	12.14	4.4	2.7
Prunjepolder (5)	1**	40*	400*	(.01)	(2)	(8)	100*	(40)	12.14	5.6	2.0
Oude Korendijk	18	8	300*	(.01)	(36)	2*	1600*	(2)	12.14	2.9	1.7

Table 5–2 Summary of the empirical data. Possible error is indicated by asterisks and brackets as follows: no asterisk – possible error less than 10%, one asterisk –30%, two asterisks –50% brackets – order of magnitude estimate

vious, or to have hydraulic properties similar to that of the covering aquitard. No data are available on the horizontal hydraulic conductivity, K_1, of the covering aquitard.

It should be noted that some of these estimates are only required to determine the flow type, as will be shown in the following chapter. In many cases they are sufficiently reliable for this purpose, even when they involve a large possible error.

6 – THE RELATIONSHIP BETWEEN THE THEORETICAL AND THE EMPIRICAL RESULTS

6.1 – *Introduction*

In the foregoing chapter (chapter 5) the empirical data from a number of different sites have been described, and have been largely summarized in table 5–2. These data consist of both information on the geohydrological properties of the formations determined by methods other than those utilizing periodic fluctuations, and of data on the propagation of periodic fluctuations through the formations. This chapter is concerned with an analysis of the relationship between these data and the theory of the propagation of periodic motions presented in chapters 2 and 3.
The purpose of this chapter is twofold. The first and more important question to be considered is: to what extent can the theory account for the flows encountered in practice? The second and closely related question is: how can the theory best be applied in the practical problem situations encountered by the geohydrologist? In this chapter an answer to both these questions will be attempted.
The relatively small number of empirical cases presented (eight sites in all) do not of course allow for an exhaustive check of the theory. They do however indicate the major points of agreement and of discrepancy between the theoretical and the empirical results, and serve to illustrate the application of the theory.

6.2 – *Short review of the theory*

6.2.1. – *The form of the potential distribution*

In chapter 2 a general form of the potential distribution has been given for the case of a simple sinusoidal motion under certain assumptions. This form is (equations 2–59):

$$h_j\,(x,z,t) = g_j(z)f(x,t) \qquad 6\text{–}1$$

where the subscript "j" refers to the number of the layers as given in the hydrostratigraphic model (figures 2–1, 3–1, 5–1).
The form of the potential distribution expressed by equation 6–1 and the further theory based on it are applicable for points sufficiently far away from the boundaries if the aquifer (layer 2 in this case) is thin, satisfying the condition (2–52):

$$\omega S_2 c_2 < 1 \qquad 6\text{–}2a$$

and if the deviations from $g_j(z)$ of the vertical potential distributions at the boundaries are not large. The region far away from a boundary at say $x = x_1$ is described by the condition (2–51):

$$x - x_1 > (K_2 D_2 c_2)^{\frac{1}{2}} \qquad 6\text{–}2b$$

where x increases away from the boundary. This condition can be relaxed if the vertical potential differences in the aquifer are very small, i.e. if $g_j(z)$ is very nearly a constant. In fact for such cases equation is likely to be applicable even very near the boundaries.

6.2.2 – *The horizontal potential distribution*

The horizontal potential distribution f(x,t) has the form (equation 2–17):

$$f(x,t) = (A_1 e^{px} + A_2 e^{-px}) e^{i\omega t} \qquad 6\text{–}3a$$

where

$$p = n + im \qquad 6\text{–}3b$$

The propagation parameters n and m are real numbers with n defined to be always positive. They can be determined through measurement of the potential fluctuations in a row of observation points, as described in chapter 4, sections 4.1 and 4.4. However the form of f(x,t) as given in equation 6–3a is itself a theoretical prediction which should be empirically checked. Such a check is best carried out for the case were there is only one (straight-line) source of the fluctuation, and no reflection effects, for then equation 6–2 reduces to the simpler form (with x increasing away from the source):

$$f(x,t) = A_2 e^{-px} e^{i\omega t} \qquad 6\text{–}4a$$

$$= A_2 e^{-nx} e^{i(\omega t - mx)} \qquad 6\text{–}4b$$

The amplitude then decreases exponentially with x, and the phase lag (mx) increases linearly with x.

6.2.3. – *The equations for the propagation parameters*

In this section some of the more important equations relating the propagation parameters n and m to the geohydrological properties of the formations will be reviewed. The validity and applicability of these equations are the main concern of this chapter.

Detailed solutions for the potential distributions and the propagation parameters are given in chapter 3. Some of the equations for n and m are given here for the special case when layer 1 is an aquitard. Particular attention is paid to the various conditions which must be satisfied for a particular set of equations to be applicable. These conditions form an important part of the theory as developed in chapters 2 and 3, and their relevance must be checked on the basis of the empirical results.

In the following summary the symbols "<<" and ">>" mean "small as compared to" and "large as compared to", and will be taken to mean that the quantities being compared differ by at least a factor 10.

The equations and conditions for n and m are enumerated below. For a detailed discussion of their derivation and significance the reader is referred to chapter 3.

a) General conditions

The following general conditions have to be satisfied in all cases:

(i) (layers 1 and 3 aquitards):

$$\omega S_1'/K_1 >> |p^2| \qquad 6\text{–}5a$$

$$\omega S_3'/K_3 >> |p^2| \qquad 6\text{–}5b$$

(ii) (layer 2 "thin" and layer 3 "thick"):

$$\omega S_2 c_2 < 1 \qquad 6\text{–}6a$$

$$\omega S_3 c_3 >> 1 \qquad 6\text{–}6b$$

b) Confined flow

The flow is confined if the following condition is satisfied:

(i) (layer 1 thick):

$$\omega S_1 c_1 >> 1 \qquad 6\text{–}7$$

If in addition the following conditions hold:

(ii) (vertical potential gradients in layer 2 negligible):

$$3c_2(\omega S_1'K_1')^{\frac{1}{2}} << 1 \qquad 6\text{–}8a$$

$$3c_2(\omega S_3'K_3')^{\frac{1}{2}} << 1 \qquad 6\text{–}8b$$

then the following equations are applicable:

$$n^2 - m^2 = \frac{1}{K_2 D_2} \left[(\omega S_1' K_1'/2)^{\frac{1}{2}} + (\omega S_3' K_3'/2)^{\frac{1}{2}} \right] \qquad 6\text{–}9a$$

$$2nm = \frac{\omega S_2}{K_2 D_2} + \frac{1}{K_2 D_2} \left[(\omega S_1' K_1'/2)^{\frac{1}{2}} + \right.$$

$$\left. + (\omega S_3' K_3'/2)^{\frac{1}{2}} \right] \qquad 6\text{–}9b$$

If in addition to conditions 6–7 and 6–8 the following condition holds:
(iii) (effect of flow in aquitards negligible):

$$\omega S_2 \gg (\omega S_1' K_1'/2)^{\frac{1}{2}} + (\omega S_3' K_3'/2)^{\frac{1}{2}} \qquad 6\text{–}10$$

then equations 6–9 reduce to:

$$n = m \qquad 6\text{–}11a$$

$$2nm = \frac{\omega S_2}{K_2 D_2} \qquad 6\text{–}11b$$

c) Semiconfined and unconfined flow

If the following condition is satisfied the flow is not confined:
(i) (layer 1 "thin"):

$$\omega S_1 c_1 < 1 \qquad 6\text{–}12$$

If the following conditions hold:

(ii) $$S_0 \gg S_1 + S_2 \qquad 6\text{–}13a$$

$$K_1 D_1 \ll K_2 D_2 \qquad 6\text{–}13b$$

(flow in layer 3 negligible):

$$\omega S_0 \gg (1 + \omega S_0 c_1)\ (\omega S_3' K_3')^{\frac{1}{2}} \qquad 6\text{–}13c$$

(vertical gradients in layer 2 small):

$$(\omega S_0 c_2)^2 < 1 + (\omega S_0 c')^2 \qquad 6\text{–}13d$$

where $c' = c_1 + c_2/3$ as defined in previous chapters, then the following equations for the general case of semiconfined or unconfined flow hold:

$$n^2 - m^2 \doteq \frac{\omega^2 S_0^{\,2} c'}{K_2 D_2 (1 + \omega^2 S_0^2 c'^2)} \qquad 6\text{–}14a$$

$$2nm = \frac{\omega S_0 (1 + \omega S_0 c')(\omega S_1 c_1/3 + \omega S_2 c')}{K_2 D_2 (1 + \omega^2 S_0^{\,2} c'^2)} \qquad 6\text{–}14b$$

If in addition to conditions 6–12 and 6–13 the following conditions hold:

(iii) $\omega S_0 c' \gg 1$ 6–15a

$\omega S_0 c' (\omega S_1 c_1/3 + \omega S_2 c') \gg 1$ 6–15b

then the flow is SEMICONFINED and equations 6–14 reduce to:

$$n^2 - m^2 = \frac{1}{K_2 D_2 c'} \qquad 6\text{–}16a$$

$$2nm = \frac{\omega S_1 c_1/3 + \omega S_2 c'}{K_2 D_2 c'} \qquad 6\text{–}16b$$

Note: If the water table is held fixed, as by drainage, then S_0 can be taken as very large and conditions 6–15 always hold.

If in addition to conditions 6–12 and 6–13 the following conditions hold:

(iv) $\omega S_0 c'(\omega S_1 c_1/3 + \omega S_2 c') \ll 1$ 6–17

then the flow is UNCONFINED and equations 6–14 reduce to:

$$n^2 - m^2 = \frac{\omega^2 S_0^{\,2} c'}{K_2 D_2 (1 + \omega^2 S_0^{\,2} c'^2)} \qquad 6\text{–}18a$$

$$2nm = \frac{\omega S_0}{K_2 D_2 (1 + \omega^2 S_0^{\,2} c'^2)} \qquad 6\text{–}18b$$

These conditions and equations provide a short review of some of the more important results of chapter 3. A particular pair of equations for n and m is only applicable in a given situation if all the corresponding conditions are satisfied. The quantities involved in the conditions can be estimated from the results of drilling and other aquifer tests. The empirical cases that are presented in this chapter function in part as examples of how the above conditions and equations can be applied in actual cases.

6.2.4 – *The vertical potential distribution*

The vertical potential distribution $g_j(z)$ has the form (equation 2–19):

$$g_j(z) = B_{1j}e^{q_jz} + B_{2j}e^{-q_jz} \qquad 6\text{–}19$$

For aquitards defined as by conditions 6–5, q_j is given by (equation 2–55):

$$q_j^2 = i\omega S_j'/K_j' \qquad 6\text{–}20$$

For thick aquitards defined by a condition such as 6–6 either B_{1j} or B_{2j} in equation 6–19 is very small as compared to the other, and the vertical potential distribution than takes a particularly simple form, which can in principle be checked empirically. Two difficulties usually stand in the way of such a check however. For one, the low hydraulic conductivities of aquitards make for special difficulties in the measurement of changes of hydraulic potentials. Also most naturally occurring aquitards are not homogenous (as assumed in the derivation of equation 6–19), but strongly layered (de Ridder and Wit, 1963; Wolff and Papadopulos, 1972). No data on periodic potential fluctuations within aquitards are available to the author, so that the validity of equations 6–19 and 6–20 cannot be directly checked.

For confined flow the vertical potential distribution within the aquifer, $g_2(z)$, reduces to a constant (put equal to unity) when conditions 6–8 hold. When conditions 6–8 hold less strongly (a "$<$" sign substituted for the "$<<$" sign) $g_2(z)$ is a quadratic function of z, if the aquifer is homogeneous. In practice the vertical hydraulic resistance of the aquifer, c_2, is usually mainly due to a few thin semipervious layers within the aquifer, and the vertical potential differences, if any, are then largely concentrated across these layers.

For semiconfined flow and especially for unconfined flow vertical flow from or to the water table is significant, and the vertical potential gradients within the aquifer are often important. In fact, when the flow is unconfined and the conditions for equations 6–18 are satisfied, then the vertical potential distribution within the aquifer is given by:

$$g_2(z) = g_2(z_2)\left\{1 - \frac{\omega S_0 c_2 (\omega S_0 c' + i)(z - z_2)^2}{(1 + \omega^2 S_0{}^2 c'^2)\, 2D_2{}^2}\right\} \qquad 6\text{–}21$$

This equation will be applied at a later point in this chapter to the case of vertical potential differences in a water table aquifer.

6.3 *Evaluation of the theory on the basis of the empirical data*

6.3.1 – *Points of comparison between theoretical and empirical results*

The short review of the theoretical results given in the previous section indicates some of the major points at which the theoretical prediction can be empirically checked.
Firstly there is the general form of the potential distributions as given in equations 6–3a and 6–19. The equations for n and m are based on these forms of the potential distributions. Thus the first point on which the theory must be checked is whether the observed potential distributions are as expected by theory. In practice reliable observations on the vertical potential distributions are difficult to obtain, and the empirical check is largely restricted to the horizontal potential distribution. If the horizontal potential distribution is of the form given in equations 6–3a or 6–4 then the horizontal propagation parameters n and m are well defined and can be determined from the measurements.
The equations for n and m can be checked on several points The frequency dependence of n and m as determined through measurements can be compared with the theoretical prediction. In addition, the theoretical relation of n and m to the geohydrological properties of the formations can be checked if these properties are known from other types of aquifer tests. Also it should be possible to verify through the empirical results whether the various conditions which describe the ranges of validity of the equations for n and m are indeed reliable.
The empirical data that are presented have been selected with an eye to obtaining a wide variety of flow types, frequencies, and stratigraphies, in order to check the theory and illustrate its application for a wide range of situations.
It should be mentioned that other authors have also reported such comparisons of theory and empirical results for the propagation of periodic fluctuations. Among these are: Timmers (1955), Wesseling (1960), van Eyden et al (1963), de Ridder and Wit (1965), Trupin (1969), Wesseling and Colenbrander (1961), and Carr (1971).

6.3.2 – *The horizontal potential distribution*

The horizontal potential distributions observed in practice often do not agree very well with the form expressed by equation 6–4b which states:

$$f(x,t) = A_2 e^{-nx} e^{i(\omega t - mx)} \qquad 6\text{–}22$$

Such discrepancy is for instance evident in the results reported by Wesseling (1960), and Carr (1971), or for the upper aquifer at Cap Pele (see figures 5–7). The discrepancies can usually be accounted for by the curvature of the shoreline, or by changes in the geohydrological properties of the formation, as is the case in the upper aquifer at Cap Pele. Where such effects can be expected to be small the observed potential should conform to equation 6–22 or 6–2.

The Zandvoort site as described in chapter 5 has a very straight shoreline and a uniform hydrostratigraphy, and therefore qualifies as a good test case for equation 6–22. The variations with distance from the shoreline of the amplitude and phase lag of the semidiurnal tidal component are illustrated in figure 5–5. These empirical results agree very well with the form of equation 6–22. Over a distance of 1760 meters (with the relative amplitudes going from 1.00 to 0.066 and the phase lags from 0.00 radians to 1.55 radians) the plots of phase lag and ln (amplitude) as given in figure 5–5 follow almost exactly the straight lines predicted by equation 6–22. Van Eyden et al (1963) found agreement with the form of equation 6–22 for two sand aquifers in the south-western Netherlands, and Wesseling and Colenbrander (1961, 1972) report similar results for the (Dalem) Tielerwaard site. These results indicate that equation 6–22 and by extrapolation equation 6–2 indeed describe the horizontal distribution if the conditions of a straight and long source and uniform hydrostratigraphy are satisfied.

The propagation parameters n and m as given in table 5–2 have been calculated on the basis of equation 6–22 or 6–2. The meaningfulness of the figures for n and m thus arrived at is also indirectly confirmed if further calculations based on them give meaningful results. That such is indeed the case will become evident in the next sections.

Since the vertical potential distributions depend on the flow types, they will be discussed together with the analysis of the equations for n and m for each flow type separately. For the few cases where data on the vertical potential distribution are available, a comparison with the theoretical prediction will be made.

6.3.3 – *Determination of the flow type*

The first step in the analysis of data on the propagation of periodic fluctuations is to determine which, if any, of the equations for n and m are applicable to the given situation. Thus, first of all, the general conditions for the applicability of the theory must be checked (conditions 6–5 and 6–6), and the flow type must be determined.
Condition 6–5a states:

$$\omega S_1'/K_1 >> |p^2| \qquad 6\text{–}23$$

For the sites described in chapter 5 this condition probably holds in every case. An exact check is not possible since little or no data on the horizontal hydraulic conductivity K_1 of the aquitard are available. But one might consider for example the case of the data for the Olst site, where the frequency is very low, and condition 6–23 is least likely to be satisfied. Assuming $K_1 = 10K_1'$ gives (see table 5–2 for the Olst data):

$$\begin{aligned}
\text{(Olst)}\quad \omega S_1'/K_1 &= 0.014 \times 2 \times 10^{-4}/0.1 = \\
&= 2.8 \times 10^{-5}\ m^{-2} \\
|p^2| &= (1.6^2 + 1.02^2) \times 10^{-6} = \\
&= 3.7 \times 10^{-6}\ m^{-2}
\end{aligned}$$

Condition 6–23 is therefore probably satisfied for the case of the periodic fluctuations at Olst. For the other sites with higher frequencies condition 6–23 certainly holds unless K_1 is improbably large. No data for the bottom aquitard layer 3 are available, but since K_3 is, for most or all of the cases, probably smaller than K_1 condition 6–5b is probably satisfied. Thus condition 6–5, defining layers 1 and 3 to be aquitards, may be safely assumed to hold.
Condition 6–6b defining layer 3 to be thick cannot be checked for lack of data on the geohydrological properties of layer 3, but it probably holds for most or all of the sites. In any case further analysis will be carried out on the assumption that this condition is satisfied, although some error may be introduced on this account, particularly for the cases of Olst and Oude Korendijk.
The aquifer, layer 2, must be thin, i.e. condition 6–6a which states that the quantity $\omega S_2 c_2$ is less than unity must be satisfied. Values of $\omega S_2 c_2$ calculated from the data given in table 5–2 are tabulated in table 6–1. Condition 6–6a is satisfied for all the sites except Zandvoort, where c_2 increases with distance away from the sea (see section 5.3), and the theory

may not be applicable for the observation points furthest away from the sea.

The restriction to points far away from the shoreline as expressed by condition 6–2b can probably be ignored for the sites at which $\omega S_2 c_2$ is very small (see section 6.2.1). It may be important however for the Prunjepolder sites and the Zandvoort site since the relatively large values of $\omega S_2 c_2$ at these sites implies that the vertical potential differences in the aquifers near the shoreline may be large at these sites. The restriction may also be important for the upper (water table) aquifer at Cap Pele, since the vertical potential differences due to vertical flow to or from the water table in this aquifer are likely to be significant. For all the sites the data for n and

SITE	FREQ ω (rads /day)	$\omega S_2 c_2$	$\omega S_1 c_1$	$\omega S_0 c'$	$(\omega S_0 c')$. $(\omega S_1 c_1'/3$ $+\omega S_2 c')$	FLOW TYPE	$n^2 - m^2$ (10^{-6} m^{-2})	2nm (10^{-6} m^{-2})
Zandvoort	12.14	<3.9	(41)	(129)	(4400)	C	1.88	2.79
,,	6.07	<1.9	(21)	(65)	(1100)	C	0.70*	1.43*
Tielerwaard (Hellouw)	24.28	(.29)	35*	(195)	(8000)	C	3.7*	10.2*
,,	12.14	(.15)	17*	(97)	(2000)	C	1.5*	6.4
Tielerwaard (Dalem)	12.14	(.10)	29*	(122)	(3000)	C	2.7	8.1
,,	0.225	(.002)	0.54*	(22.5)	(10.2)	S	0.64*	0.26*
Cap Pele (-upper)	12.14	.18**	0	4.6**	0.28**	U	100	62
,,	6.07	.09**	0	2.3**	0.69**	U	70*	48*
Cap Pele (lower)	12.14	(.07)	(73)	(18000)	5.0×10^5	C	124	115
,,	6.07	(.03)	(36)	(9000)	1.3×10^5	C	68*	66*
Borden	12.14	(.09)	(6.1)	(63)	(180)	C or S	-11*	129
,,	6.07	(.04)	(3.0)	(31)	(45)	C or S	-13*	72
Olst	0.028	(2.5×10^{-5})	(.002)	(0.56)	(.0014)	U	4.4**	2.8**
,,	0.014	(1.3×10^{-5})	(.001)	(0.28)	(.0034)	U	1.5*	3.3*
Prunjepolder (2)	12.14	(.39)	(15)	(93)	(1100)	C	12.3*	24.0
Prunjepolder (5)	12.14	(.39)	(0.24)	(14)	(16)	S	27*	22*
Oude Korendijk	12.14	(.005)	(70)	(194)	(5300)	C	5.5	9.9

Table 6–1 Determination of the flow type (C – confined flow, S – semiconfined flow, U – unconfined flow). The possible errors are indicated as for table 5–2: no brackets or asterisks – ± 10%, one asterisk – ± 30%, two asterisks – ± 50%, brackets – order of magnitude estimate.

m have been calculated on the assumption that all the observation points are sufficiently far away from the shoreline.
The flow type itself is determined by a few characteristic numbers as follows (see section 6.2.3):
The flow is confined if:

$$\omega S_1 c_1 \gg 1 \qquad 6\text{–}24$$

The flow is semiconfined if:

$$\omega S_1 c_1 < 1 \qquad 6\text{–}25a$$

$$\omega S_0 c' \gg 1 \qquad 6\text{–}25b$$

$$\omega S_0 c' (\omega S_1 c_1/3 + \omega S_2 c') \gg 1 \qquad 6\text{–}25c$$

The flow is unconfined if:

$$\omega S_1 c_1 < 1 \qquad 6\text{–}26a$$

$$\omega S_0 c' (\omega S_1 c_1/3 + \omega S_2 c') \ll 1 \qquad 6\text{–}26b$$

The three characteristic numbers appearing on the left hand side of these conditions can be calculated from the data for the geohydrological characteristics of the formations obtained from tests other than those involving periodic fluctuations, as tabulated in table 5–2. The results of this calculation are summarized in table 6–1 together with the flow types as determined by the above conditions. Because of the possible error involved in the estimates of the characteristic numbers, the decision as to the flow type is doubtful in some cases. For the two cases of semiconfined flow for instance (Tielerwaard, Dalem, $\omega = 0.225$ rads/day; and Prunjepolder 5) it is possible that condition 6–25c is in fact not fully satisfied. Analysis of the data for periodic flow can yield an additional check on the flow type in some cases as will be shown at a later point.
For the cases of Borden, and strictly speaking, for the upper aquifer at Cap Pele with $\omega = 12.14$ rads/day, the flow type does not fall into one of the three categories that are used. For the Borden case further analysis of the data for periodic flow cannot resolve the question of flow type, as will be come apparent later. For the Cap Pele case the flow type will for the present be assumed to be simply confined. The results of the periodic flow analysis on this basis will indicate whether this assumption is in fact correct. Here, as

elsewhere at further points, the most simple possible case is assumed to hold until further evidence is presented.
Also given in table 6–1 for further reference are the quantities $n^2 - m^2$ and 2nm, since these crop up repeatedly in the equations for n and m.
A number of subsidiary conditions must also hold if the special equations for n and m for the various flow types are to be applicable. These conditions will be considered in connection with the analysis for the different flow types, as given in the following sections.
As has been mentioned before, the criteria for the different flow types as given above are themselves theoretical results whose validity will be checked on the basis of the empirical results.

6.3.4. – *Confined flow*

In this section the theory for confined periodic flow as summarized by conditions and equations 6–7 through 6–11 will be evaluated on the basis of the empirical results. The relevant data and the results of the various calculations are summarized in table 6–2. Data for some of the cases of semiconfined and unconfined flow are also included as part of the investigation as to whether condition 6–7 (or 6–24) does indeed describe the range of validity of the equations for confined flow.
(a) Subsidiary conditions
For equations 6–9 or 6–11 for confined flow to hold, conditions 6–8 relating to the importance of vertical potential gradients in the aquifer must hold. Since no data for layer 3 are available only condition 6–8a which states:

$$3c_2(\omega S_1' K_1')^{\frac{1}{2}} << 1 \qquad 6\text{–}27$$

will be investigated. The quantity $S_3'K_3'$ is probably not greater than $S_1'K_1'$, and thus condition 6–8b holds if condition 6–27 (or 6–8a) holds. In column 4 of table 6–2 the quantity $3c_2\sqrt{(\omega S_1'K_1')}$ is tabulated as calculated from the data for the geohydrological properties of the formations as given in table 5–2. These figures indicate that condition 6–27 is possibly not satisfied for some of the cases of confined flow.
For the case of Zandvoort data on the vertical variation of potential within the aquifer are available. They are tabulated in table 5–1 in terms of the variation with height in the aquifer of the amplitude and the phase of the semidiurnal tidal component. These data indicate that within the upper part of the Zandvoort aquifer (20 to 80 meters below sea level) there is little or no vertical variation of potential. Now the quantity $3c_2\sqrt{(\omega S_1'K_1')}$

for the upper part of the aquifer only, can be estimated with the assumption $K_2' = 0.1K_2$ to be about 0.3, so that indeed according to condition 6–27 little vertical variation of potential would be expected. On the other hand at x = 1760 meters, the loam layers separating the upper and lower parts of the aquifer probably have significant vertical resistance (up to 200 days) so that at this point the value of $3c_2\sqrt{(\omega S_1 K_1')}$ for the whole aquifer may approach the maximum values indicated in table 6–2. The data of table 5–1 do indicate some vertical variation of potential at x = 1760 m. These data for the Zandvoort aquifer would seem to indicate that conditions 6–8 do indeed describe the case of negligible vertical potential variations.
For the present the analysis will be continued on the assumption that conditions 6–8 are satisfied for all the cases of confined flow, according to the principle of trying the least complicated possibility first. An equation for confined periodic flow including the effect of vertical potential gradients in the aquifer has been given in chapter 3 (equation 3–20), so that an analysis including these effects is possible if necessary.
If condition 6–10 which can be written

$$\frac{(S_1'K_1')^{\frac{1}{2}}}{S_2\ (2\omega)^{\frac{1}{2}}} + \frac{(S_3'K_3')^{\frac{1}{2}}}{S_2\ (2\omega)^{\frac{1}{2}}} \ll 1 \qquad 6\text{–}28$$

is satisfied then the effects of flow in the aquitards is expected to be negligible and equations 6–11 hold. The quantity $\sqrt{(S_1'K_1'/2\omega)}S_2^{-1}$ is tabulated in column 5 of table 6–2, as calculated from the data in table 5–2. These figures indicate that condition 6–28 is in no case clearly satisfied. It would seem therefore that for the analysis of the data for n and m equations 6–9, which include the effect of flow in the aquitards, must be applied.

(b) The equations for n and m

Equations 6–9 can be written in the form:

$$(S_1'K_1')^{\frac{1}{2}}/K_2D_2 = (n^2 - m^2)/(\omega/2)^{\frac{1}{2}} \qquad 6\text{–}29a$$

$$S_2/K_2D_2 = (2nm - n^2 + m^2)/\omega \qquad 6\text{–}29b$$

Here the important and risky assumption has been used that the lower aquitard is impervious, partly for the simple reason that no independent data for this aquitard are available. Equation 6–29b is not effected by this assumption, but the value of $\sqrt{(S_1K_1')}/K_2D_2$ calculated with equation 6–29a from the

measured values of n and m may be too large for cases where the flow in layer 3 may not be negligible.
Equations 6–9 have been rewritten in the form of equations 6–29 because the right hand sides of equations 6–29 should be independent of frequency,

1	2	3	4	5	6	7
SITE	FREQ ω (rads /day)	FLOW TYPE	$3c_2 \cdot (\omega S_1' K_1')^{\frac{1}{2}}$	$\left(\frac{\omega S_1' K_1'}{2\omega}\right)^{\frac{1}{2}} / S_2$	$(S_1' K_1')^{\frac{1}{2}} / K_2 D_2$ ($10^{-7} m^{-2}\ d^{+\frac{1}{2}}$) periodic	other
Zandvoort	12.14	C	<(3.9)	(.24)	7.7	(4.1)
,,	6.07	C	<(2.7)	(.33)	4.0*	,,
Tielerwaard (Hellouw)	24.28	C	(.18)	.14*	10.6*	6.0*
	12.14	C	(.13)	.20*	5.8*	,,
Tielerwaard (Dalem)	12.14	C	(.11)	.26*	11.1	7.7*
Cap Pele (lower aquifer)	12.14	C	(.46)	(1.7)	502	(1200)
	6.07	C	(.33)	(2.5)	390*	,,
Prunjepolder (2)	12.14	C	(.61)	(3.7)	50*	(37)
Oude Korendijk	12.14	C	(.03)	(1.4)	22.4	(50)
Tielerwaard (Dalem)	0.225	S	(.015)	(1.9)	19.3**	7.7*
Borden	12.14	C or S	(.81)	(2.2)	-42*	(3500)
,,	6.07	C or S	(.57)	(3.1)	-74*	,,
Prunjepolder (5)	12.14	S	(.59)	(3.6)	111*	(35)
Cap Pele (upper aquifer)	12.14	U	–	–	402	0
,,	6.07	U	–	–	393*	0
Olst	0.028	U	–	–	374**	(18)
,,	0.014	U	–	–	182*	,,

Table 6–2 Data for the evaluation of the confined flow conditions and equations on the basis of the empirical results. The possible error and the flow type are indicated as in Table 6–1.

and involve only data related directly to the propagation of the periodic fluctuations. A good test of the theory can thus be carried out with equations 6–29.
The quantities $\sqrt{(S_1'K_1')}/K_2D_2$ and S_2/K_2D_2 as calculated with equations

1	8	9	10	11	12
SITE	S_2/K_2D_2 ($\times 10^{-7}$ d m^{-2})		$2nm/\omega$ ($\times 10^{-7}$d. m^{-2})	$\omega S_1 c_1$	
	periodic	other		periodic	other
Zandvoort	0.74*	3.6*	2.3	143**	(41)
,,	1.2*	,,	2.4*	20**	(21)
Tielerwaard (Hellouw)	2.7*	6.0*	4.2*	110**	35*
,,	4.1*	,,	5.3	18*	17*
Tielerwaard (Dalem)	4.4	6.0*	6.6	58*	29*
Cap Pele (lower aquifer)	-7.1**	150**	95	(12)	(73)
,,	(-3.9)	,,	108*	(3.7)	(36)
Prunjepolder (2)	9.6*	(20)	19.6	27**	(15)
Oude Korendijk	3.6*	6.7**	8.1	14**	(70)
Tielerwaard (Dalem)	-17.1**	6.0*	11.7**	3.3**	0.54*
Borden	115	(320)	106	–	(6)
,,	140	,,	119	–	(3)
Prunjepolder (5)	-4.1*	(20)	18.5*	(2.3)	(0.24)
Cap Pele (upper aquifer)	-30*	7.1*	1000	0	0
,,	-35**	,,	2400*	0	0
Olst	-577**	(4)	51**	(0.9)	(.002)
,,	1250**	,,	77**	(0.1)	(.001)

Table 6–2 (continued)

6–29 from the measured values of n and m are tabulated in columns 6 and 8 of table 6–2. For comparison the same quantities calculated from the results of other tests as given in table 5–2 are tabulated in columns 7 and 9.
For the present only the cases of confined flow will be considered. The first point to be noted is that the results from periodic flow given in columns 6 and 8 exhibit a frequency dependency which can probably not be accounted for by the possible error only. Secondly, the agreement with the results of other tests is not particularly good, especially for the case of the quantity S_2/K_2D_2, where the possible error cannot account for the discrepancies. This disagreement between theoretical and experimental results may be in

1	2	3	4	5	6	7	8
SITE	FREQ ω (rads/ /day)	FLOW TYPE	B	$\frac{S_1C_1/3 + S_2c'}{K_2D_2c'}$ ($\times 10^{-7}$ day^{2}m^{-2})		K_2D_2c' ($\times 10^5$ m^2)	
				periodic	other	periodic	other
Tielerwaard (Dalem)	0.225	S	(5×10^{-5})	11.7*	10*	15.5*	20
Prunjepolder(5)	12.14	S	(.12)	18.5*	(21)	0.37*	.45**
Tielerwaard (Dalem)	12.14	C	(5×10^{-4})	6.6	10*	3.7	20
Prunjepolder (2)	12.14	C	(3×10^{-3})	19.6	(33)	0.83*	3.1**
Zandvoort	12.14	C	(.016)	2.3	(5.9)	5.3	48**
”	6.07	C	(.016)	2.4*	”	14.4*	”
Cap Pele	12.14	U	8.6**	51.4	7.1**	0.10	.070*
”	6.07	U	7.5**	77.4*	”	0.15*	”
Olst	0.028	U	(5×10^{-5})	1005**	(5.8)	2.26**	(1.5)
”	0.014	U	(2×10^{-5})	2300*	”	6.6*	”

Table 6–3 Evaluation of the equations and conditions for semiconfined flow on the basis of empirical results. The possible error and the flow types are indicated as in table 6–1. The quantity B in column 4 equals $\omega^2 S_0{}^2 c_0{}^2 (1 + \omega^2 S_0{}^2 c'^2)^{-1}$.

part due to the effects of flow in the bottom aquitard, and of vertical potential gradients within the aquifer, but it seems unlikely that these can account for all of the observed discrepancy, as further inspection of the data in table 6–2 shows.

It would seem from these results that the reliability of equation 6–29a is at best questionable, while equation 6–29b is simply not applicable in many cases.

The possibility remains that not equations 6–9 but equations 6–11 should be applied. These equations for the case of negligible flow in the aquitards can be written:

$$n^2 - m^2 = 2nm \qquad 6\text{–}30a$$

$$S_2/K_2D_2 = 2nm/\omega \qquad 6\text{–}30b$$

1	2	3	4	5	6	7
SITE	FREQ ω (rads / /day)	FLOW TYPE	S_0c' (days)		K_2D_2c' ($x10^5m^2$)	
			periodic	other	periodic	other
Cap Pele (upper aquifer)	12.14	U	0.13	0.38**	.072	.070*
,,	6.07	U	0.25	,,	.099*	,,
Olst	0.028	U	57**	(20)	1.6**	(1.5)
,,	0.014	U	33*	,,	1.2*	,,
Tielerwaard (Dalem)	0.225	S	11*	(100)	13.3*	20
Prunjepolder (5)	12.14	S	0.10*	(1.2)	0.22*	.45**
Tielerwaard (Dalem)	12.14	C	.028	(100)	0.38	20
Zandvoort	12.14	C	.056	(11)	1.67	48**
,,	6.07	C	.079*	,,	2.74*	,,

Table 6–4 Data for the evaluation of the equations and conditions for unconfined flow. The possible error and the flow type are indicated as in Table 6–1

As was already shown, the condition 6–28 (or 6–10), which must hold if these equations are to be applicable, is probably not satisfied for the cases treated here. Inspection of the values of n^2-m^2 as given in table 6–1 shows immediately that equation 6–30a is at any rate not satisfied for the confined flow cases. Thus the previous conclusion that the effect of flow in the aquitards is not negligible would seem to be corroborated.
In column 10 of table 6–2 the quantity $2nm/\omega$ is tabulated. It can be seen that this quantity turns out to be very nearly independent of frequency, as would be expected if equation 6–30b holds. In addition comparison with the values of S_2/K_2D_2 calculated from other tests, as given in column 9, shows good agreement in every case. Judging by the empirical data then, it would seem that equation 6–30b may well be generally valid for confined flow.
These empirical results are difficult to account for theoretically. It may well be that the strongly layered character of the aquitards is significant, and that the aquitards cannot be assumed homogeneous as was done in the development of the theory given here. It is also possible that the release of water from elastic storage is not exactly proportional to the changes of hydraulic potential, but involve a "hysteresis" effect that cannot be neglected.

(c) The criteria for confined flow

The relevance of the condition for confined flow (condition 6–7):

$$\omega S_1 c_1 >> 1 \qquad 6\text{–}31$$

can also be checked on the basis of the data given in table 6–2.
In column 11 of table 6–2 are given values of $\omega S_1 c_1$ calculated from the data for n and m through a modified form of equation 6–29a:

$$\omega S_1 c_1 = 2(n^2-m^2)^2(K_2D_2c_1)^2 \qquad 6\text{–}32$$

The results of this calculation for the confined flow cases agree to some extent with the values of wS_1c_1 calculated from the results of other tests, as given in column 12, and all satisfy condition 6–31. To this extent the theory is consistent even if the reliability of equation 6–32 is somewhat in doubt.
For the cases of semiconfined and unconfined flow the values of wS_1c_1 calculated with equation 6–32 do not satisfy condition 6–31 for confined flow. Thus it would seem possible to distinguish between confined flow and not confined flow partly on the basis of the measured values of n and m, but such a method of determining the flow type is probably not wholly reliable.

If condition 6–31 does indeed define confined flow, then equations 6–29, applied to the data for semiconfined and unconfined flow, should give results in disagreement with the results from other texts. For unconfined flow this conclusion is certainly born out, as inspection of the relevant data in table 6–2 will show immediately. For semiconfined flow this disagreement is less striking, especially in view of the poor agreement for confined flow itself. The analysis of semiconfined flow which follows will yield stronger evidence for the validity of condition 6–31 as a definition of confined flow.

6.3.5 – *Semiconfined flow*

Data relevant to the evaluation of the theory for semiconfined flow are tabulated in table 6–3. Only two cases of semiconfined flow are available. Data for some of the cases of confined flow and of unconfined flow are included in connection with the evaluation of the criteria for semiconfined flow.

(a) Subsidiary conditions for semiconfined (and unconfined) flow

In addition to the principal conditions for the flow types (conditions 6–25 and 6–26) a number of subsidiary conditions must be satisfied if the equations involving n and m for semiconfined and unconfined flow are to be applicable.

Condition 6–13a states:

$$S_0 \gg S_1 + S_2 \qquad \text{6–33a}$$

This condition is probably satisfied for all the cases of semiconfined and unconfined flow, as inspection of the data for S_0, S_1, and S_2 in table 5–2 will show.

Condition 6–13b states:

$$K_1 D_1 \ll K_2 D_2 \qquad \text{6–33b}$$

No data for K_1 are available, but this condition is probably easily satisfied for all the cases that are treated.

Condition 6–13c states that flow in layer 3 is negligible if:

$$\omega S_0 \gg (1 + \omega S_0 c_1) \surd(\omega S_3' K_3') \qquad \text{6–33c}$$

Some data for layer 3 are only available for the case of the upper aquifer at Cap Pele. The data given in table 5–2 give (layer 3 for the upper aquifer is the same as layer 1 for the lower aquifer):

$$\omega S_0 = (.015)(12.14)$$
$$= 0.18 \text{ day}^{-1}$$

$$(1 + \omega S_0 c_1)\sqrt{(\omega S_3' K_3')} = (1+0)\sqrt{(12.14 \times 2 \times 10^{-6} \times 3 \times 10^{-4})}$$
$$= 8.5 \times 10^{-5} \text{ day}^{-1}$$

Thus at least for this case condition 6–33c (or 6–13c) is easily satisfied. For the other cases it will be assumed that condition 6–33c holds, although there is some doubt as to the validity of this assumption, especially for the case of Prunjepolder 5, where layer 3 may be fairly permeable.
Condition 6–13d relates to the vertical potential gradients in the aquifers. It states:

$$(\omega S_0 c_2)^2 / (1 + (\omega S_0 c')^2) < 1 \qquad \text{6–33d}$$

The quantity $(\omega S_0 c_2)^2 / (1 + \omega^2 S_0{}^2 c'^2)$ is tabulated in column 4 of table 6–3. It turns out that condition 6–33d is satisfied for all cases except that of the upper aquifer at Cap Pele. The special problem of the unconfined flow in this water table aquifer will be discussed in connection with the analysis of the theory for unconfined flow. For the cases of semiconfined flow condition 6–33d gives no difficulties.

(b) The equations for n and m

Equations 6–14 relating n and m to the geohydrological properties of the formations for the case of semiconfined flow can be written:

$$(S_1 c_1/3 + S_2 c')/K_2 D_2 = 2nm/\omega \qquad \text{6–34a}$$

$$K_2 D_2 c' = (n^2 - m^2)^{-1} \qquad \text{6–34b}$$

Here again the right hand sides of these equations involve only data on the propagation of periodic fluctuations, and are independent of frequency. The results for the quantities $(S_1 c_1/3 + S_2 c')/K_2 D_2$ and $K_2 D_2 c'$ as calculated by equations 6–34 are tabulated in columns 5 and 7 of table 6–3. For the semiconfined flow cases these values agree well with the values calculated from other tests, as tabulated in columns 6 and 8.
Not enough data are available for a thorough check, but it would seem that equations 6–34 (or 6–16) may well be reliable.

(c) The criteria for semiconfined flow
Since equations 6–34 give results for the geohydrological properties of the formations in good agreement with the results of other tests, these results for periodic flow will also be consistent with figures by which the flow type was decided, as given in table 6–1.
If conditions 6–25 do indeed delimit the cases of semiconfined flow, then equations 6–34 applied to data for confined or unconfined flow should give results in disagreement with those of other tests. The relevant data for checking this conclusion are given in table 6–3.
For confined flow the quantity K_2D_2c' calculated with equation 6–34b shows clear disagreement with the results of other tests. This discrepancy might be expected, for in fact equation 6–32 for confined flow can be written:

$$K_2D_2c_1 = \sqrt{(\omega S_1c_1/2)}\,(n^2-m^2)^{-1} \qquad 6\text{–}35$$

For nearly all cases of confined and semiconfined flow $c_1 = c'$. For each case of confined flow the quantity K_2D_2c' calculated by equation 6–34b is in fact too small by a factor approximately equal to $\sqrt{(\omega S_1c_1/2)}$. This result confirms the conclusion that confined flow is defined by the condition that ωS_1c_1 be much greater than one.
For unconfined flow the value of $(\omega S_1c_1/3 + \omega S_2c')/K_2D_2$ calculated with equation 6–34a is far too large, as inspection of the figures in columns 5 and 6 of table 6–3 shows. It may be concluded that the criteria for semiconfined flow are also valid as a distinction between semiconfined and unconfined flow.

6.3.6 – *Unconfined flow*

Data relevant to the evaluation of the equations and conditions for unconfined flow are given in table 6–4. As before data for other flow types are included as part of the empirical check on the criteria for unconfined flow.
(a) The subsidiary conditions
In addition to the conditions for unconfined flow (conditions 6–26) several subsidiary conditions must be satisfied if equations 6–18 for n and m in the case of unconfined flow are to be applicable. These are the same subsidiary conditions as for semiconfined flow, and have been discussed in the previous section on semiconfined flow. Condition 6–13d concerning the importance of vertical potential gradients within the aquifer was found not to be satisfied for the case of the upper (water table) aquifer at Cap Pele. For the moment

however the analysis of the equations for n and m will be continued as if condition 6–13d gave no difficulty. Analysis of the data for the vertical potential gradients in the aquifer at Cap Pele will show that condition 6–13d is in fact satisfied because the assumption that $c_2 = 3c'$ and $c_1 = 0$ is not valid.

(b) The equations for n and m

Equations 6–18 relating n and m to the geohydrological properties of the formations for unconfined flow can be written:

$$S_0 c' = \frac{1}{\omega}\,\frac{n^2 - m^2}{2nm} \qquad 6\text{–}36a$$

$$K_2 D_2 c' = \frac{n^2 - m^2}{(n^2 + m^2)^2} \qquad 6\text{–}36b$$

where as before the right hand sides of these equations are independent of frequency, and involve only the data on the propagation of periodic components.

In columns 4 and 6 of table 6–4 the values of $S_0 c'$ and $K_2 D_2 c'$ are given as calculated from the measured values of n and m through equations 6–36. In columns 5 and 7 the same quantities are tabulated as calculated from the results of other tests.

For the unconfined flow cases the value of $K_2 D_2 c'$ calculated with equation 6–36b agrees well with the results of other tests, and would seem to exhibit at least approximately the expected independence from frequency.

The results for $S_0 c'$ calculated with equation 6–34, show both a frequency dependence and a disagreement with the results of other tests, the more clearly so for the Cap Pele case. As has already been mentioned in chapter 5, it seems likely that for fluctuations with durations of the order of a few days or less, the value of S_0 may be considered as increasing with the duration of the flow. The results for Cap Pele bear out this conclusion, since the fluctuation with the longer duration yields a larger value of $S_0 c'$. For the much slower fluctuations at Olst the possible error in the results leaves open the possibility that S_0 is constant and equal to the effective porosity (about 0.2).

Disregarding for the moment the question of the conditions for equations 6–36, it would seem from these results that equations 6–36 may well be reliable if the value of S_0 is considered to increase with decreasing frequency.

(c) Vertical potential gradients in the aquifer

Some empirical data are available on the vertical potential gradients within the upper aquifer at Cap Pele, (see figures 5–7a and 5–7b). These allow an evaluation of the theoretical prediction for the vertical gradients and a reassessment of condition 6–13d involving the vertical gradients.

For unconfined flow the vertical potential distribution in the aquifer is given by (equation 6–21):

$$g_2(z) = g_2(z_2) \left\{ 1 - \frac{\omega S_0 c_2 \, (\omega S_0 c' + i)}{1 + (\omega S_0 c')^2} \, \frac{(z-z_2)^2}{2D_2{}^2} \right\} \qquad 6\text{–}37$$

Divided into its real and imaginary parts, this equation indicates that from the bottom to the top of the aquifer there should be a decrease of amplitude and an increase in phase lag. The results for Cap Pele, given graphically in figures 5–7, bear out this theoretical conclusion.

Through equation 6–37 the data for the amplitudes and the phase lags in the bottom half and near the top of the aquifer can be analyzed to yield values of $S_0 c_2$ and $S_0 c'$. The data of the piezometer at x = 44 m may not be reliable because of the influence of boundary effects (this piezometer is only about 50 m away from the sea, while $\sqrt{(K_2 D_2 c_2)} = 140$ m (see condition 6–2b). For the piezometer at x = 114 m, $z' = (z-z_2)/D_2 = 0.93$, the data yield, through equation 6–37:

semidiurnal component:	$S_0 c'$	$\doteq$	0.12 day
	$S_0 c_2$	=	0.16 day
diurnal component:	$S_0 c'$	$\doteq$	0.19 day
	$S_0 c_2$	=	0.21 day

Agreement with the values of $S_0 c'$ calculated from the data for the horizontal propagation of the fluctuations (0.13 and 0.25 day for the semi-diurnal and the diurnal components respectively) is quite good. To this extent the theory seems consistent.

The results for the vertical potential variations indicate another problem however. With no covering aquitard c_1 should be zero, and then $S_0 c' =$ $= S_0(c_1 + c_2/3)$ should be equal to $S_0 c_2/3$. The results for $S_0 c'$ and $S_0 c_2$ given above do not agree with this conclusion, and seem to indicate that for some reason c_1 cannot be assumed to be zero. In fact they yield $c_1 = (0.5 \pm 0.1)\, c_2$. Essentially it has turned out that the observed vertical potential gradients are not as large as expected.

This result can be probably be interpreted as implying that there is an extra resistance to the vertical flow of water near the water table, perhaps due to the only partial saturation of the porous medium at the transition zone between the saturated and the unsaturated zones. The variation of S_0 with the duration of the flow as mentioned in the previous section may well be related to this result.

With the values of $S_0 c'$ and $S_0 c_2$ determined from the observations on the vertical potential variations, condition 6–13d, which was previously found not to be satisfied, can be reassessed, It states:

$$(\omega S_0 c_2)^2 (1 + \omega^2 S_0{}^2 c'^2)^{-1} < 1 \qquad 6\text{–}38$$

With the values of $S_0 c_2$ and $S_0 c'$ found from the vertical gradients, the quantity on the left hand side of condition 6–38 takes values of 1.2 and 0.7 for the semidiurnal and the diurnal components respectively. Thus condition 6–38 (or 6–13d) is fairly well satisfied for the case of the Cap Pele aquifer and the application of equations 6–36 for n and m is probably legitimate.

d) The criteria for unconfined flow

As was already mentioned previously the conditions for unconfined flow are perhaps not entirely satisfied for the case of the semidiurnal tidal fluctuation in the water table aquifer at Cap Pele. In fact condition 6–26b which states:

$$\omega S_0 c' (\omega S_1 c_1 / 3 + \omega S_2 c') << 1 \qquad 6\text{–}39$$

seems not to be satisfied, as the figures given in table 6–1 show. However, the analysis of the data for periodic flow yield a lower value of $S_0 c'$, in fact low enough that condition 6–39 turns out to be satisfied. The cause of this seeming contradiction lies in the fact that the value of S_0 used in the initial estimate for the left hand side of condition 6–39 was obtained from a pump test of 1200 minute duration. The effective duration of the semidiurnal component is much less than 1200 minutes, and consequently a lower value of S_0 is applicable for this component.

If the criteria for unconfined flow are correct (conditions 6–26), then application of equations 6–36 to the data for semiconfined and confined flow should give results for the geohydrological characteristics of the formations in disagreement with the results of other tests. The figures relevant to a check of this conclusion are given in table 6–4.

For the case of confined flow the values of $S_0 c'$ and $K_2 D_2 c'$ calculated with equations 6–36 are obviously erroneous. For the case of seminconfined flow

the values of $S_0 c'$ obtained with equation 6–36a are probably too low and the values of $K_2 D_2 c'$ obtained with equation 6–36b are not clearly wrong although they seem rather low.
These results confirm to some extent the conclusion arrived at in the previous section with regard to the analysis of semiconfined flow, namely that the criteria for distinguishing between semiconfined and unconfined flow (conditions 6–25 and 6–26) are correct and can be applied to practical situations. The distinction between unconfined and confined flow on this basis of these criteria is certainly valid.

6.4 – *Validity and applicability of the theory – a summary*

At the beginning of this chapter the two major points of interest in the present analysis were mentioned. These are the questions concerning the validity of the theory and its application in practical situations. In the previous section, concerned with the evaluation of the theory on the basis of the empirical results, the question as to the validity of the theory was treated explicitly. At the same time an implicit answer to the second question (concerned with applicability) has been given because the treatment of the theoretical and the empirical results gives in effect a variety of examples as to how the theory may be applied to actual situations. In essence therefore the present section consists of a review and summary of the results obtained in the previous sections.
Section 6.2 of this chapter gives a concise review of the theory for the propagation of periodic fluctuations as developed in chapters 2 and 3. This section has been included partly because it may be useful to the reader who desires a quick overall view of the theoretical results before going on to their application.
As shown in section 6.3.2 the horizontal potential distributions encountered in practice may be expected to agree with equations 6–2 or 6–4 if disturbing effects due to the source of the fluctuations or to large-scale changes in the geohydrological properties of the formations are negligible. This result implies that the horizontal propagation parameters n and m are unambiguously defined and can be used for further analysis. It also implies that even where further analysis of n and m is not possible or required, equations 6–2 and 6–4 have predictive value in themselves, and may be useful for cases where the hydraulic potentials must be calculated or predicted.
The determination of the flow type by means of conditions 6–24, 6–25, and 6–26 as described in section 6.3.3 is an essential element of the theory. The results of further analysis show that these conditions do indeed give

a valid and useful method of distinguishing between flow types. The definitions of the flow types as given in chapter 3 were as follows: the flow is confined if it is isolated from the water table; the flow is semiconfined if it is influenced by the position of the water table but the water table does not move significantly; the flow is unconfined if it is wholly determined by the position of the water table. The flow type must be determined on the basis of data obtained from independent tests. The results of analysis of the data for periodic flow can be used to check the figures by means of which the flow was decided, but cannot for most cases be used to give a reliable determination of the flow type.

Again it is emphasized that the use of quantitative criteria for determining the flow type and selecting the appropiate equations for the flow is an essential element of the theory that has been developed.

The equations relating the horizontal propagation parameters n and m to the geohydrological properties of the formations are only applicable if various subsidiary conditions summarized in the review of the theory, are also satisfied. It may happen that data required to check whether these conditions hold are not available, or that in any case it is doubtful whether they are satisfied. For cases where there is a reasonable probability that these subsidiary conditions are at least roughly satisfied, the best approach is probably to carry on with the analysis bearing in mind the possibility that the results may err. In some cases the nature of the results may indicate the extent of the error. This is essentially the approach used in this chapter.

Equations 6–9, or 6–29, describing the horizontal propagation of a sinusoidal fluctuation for the case of confined flow, have been evaluated on the basis of the empirical results from a number of different sites with different frequencies. It turns out that equation 6–9a, or 6–29a, may be reliable, but that equation 6–9b, or 6–29b, does not give good results and is probably not reliable. However, the empirical results indicate that equation 6–11b, or 6–30b, gives good results for all the cases of confined flow that have been treated, and may well be generally applicable for confined flow.

Only two empirical cases were available by which equations 6–14, or 6–34, for semiconfined flow, could be evaluated. For these two cases analysis based on equations 6–34 gave good results, which indicates that equations 6–14, or 6–34 may well be reliable.

Equations 6–18, or 6–36, describe the horizontal propagation of a sinusoidal fluctuation for the case of unconfined flow. Evaluation of these equations on the basis of empirical results indicates that equation 6–36b is probably fairly reliable, but that equation 6–36a can only be considered as reliable if the storage at the water table may be assumed to increase with the period of the flow.

Observations on the vertical variation of potential within the aquifer for unconfined flow have been analyzed with equation 6–21 (or 6–37). The results indicate that the vertical potential differences are smaller than was expected on the basis of measurements of the horizontal propagation of the fluctuations. It seems that there is an extra resistance to vertical flow near the water table, an empirical result which may well be related to the variation of the storage at the water table, as mentioned above.

On the whole than it may be concluded that except for some special problems of confined and unconfined flow the theory that has been developed is at least fairly reliable, and may be applied with some confidence in appropriate situations.

SUMMARY

The purpose of the present work is twofold: firstly, to develop a comprehensive theory for the propagation of periodic fluctuations of groundwater potential, and secondly, to evaluate this theory and illustrate its application by means of empirical results. from a number of different sites.

The theory, as presented in chapters 2 and 3, is based on Darcy's law and a linear proportionality between changes of storage and changes of hydraulic potential. It is found that for thin aquifers and points sufficiently far away from the boundaries a special solution applies, independent of the potentials at the boundaries except that they are periodic in time. Explicit expressions of this solution for various cases of confined, semiconfined and unconfined flow are given, including equations for the horizontal propagation of the waves and the vertical potential distributions in the aquifers and aquitards. The ranges of applicability of the various equations are defined in terms of quantitative criteria. Through this systematic approach equations for special cases of periodic flow previously obtained by other authors are integrated in a more general theory.

In chapter 4 methods of analyzing and predicting periodic motions are discussed, including the computation of sinusoidal components, the elimination of tidal effects from water level records, and the problem of well response. Some of these methods may well be useful, even in situations where application of the theory for the propagation of the fluctuations is not possible or necessary.

In chapter 5 empirical data from various sites in the Netherlands and Canada are presented. These include data for the propagation of periodic fluctuations due to tidal motions or changes in river stage, and data on the geohydrological characteristics of the formations obtained by pump testing and like methods.

Chapter 6 starts out with a summary of some of the more significant theoretical results. These results for confined, semiconfined and unconfined flow, including the criteria for the applicability of the various equations, are then compared with the empirical results. It turns out that the given criteria do indeed provide a meaningful and useful method of distinguishing flow types and establishing the applicability of particular equations. The equations for wave propagation under confined flow conditions are at best only partially applicable to actual situations, but an empirical relationship is found which can probably be used instead. The equations for semiconfined flow may well be generally applicable. For unconfined flow the equations for horizontal wave propagation may be valid if the coefficient of storage at the water table can be considered to increase with the period of the fluctuation. The predicted vertical potential distribution within the aquifer for

unconfined flow agrees with the measured potentials if it can be assumed that there is an extra resistance to flow near the water table.

The comparison of the theoretical and the empirical results indicate at least two problems which require further investigation. For confined flow there is a discrepancy between theory and observation which might perhaps be resolved if the assumption of homogeneity of the aquitards is dropped. For unconfined flow the present theory does not give entirely adequate account of the flow and storage at the water table.

In general it may be said that the theoretical approach that has been used, namely that of obtaining both equations for various flow conditions and criteria for their applicability from one general model, is of value both for the evaluation of the equations, and for their application to practical problems.

EPILOGUE

It seems good to conclude with the words of an early Dutch engineer and hydrologist, well-known for his work in the drainage of polders, whose attitudes, though expressed in an older language, are valid as ever.

"Hier mede wil ik mijn schrijven afkorten/ en zo ik hier inne wat gemist mochte hebben/ het welke niet zo wel getroffen is/ als in't bedijken wel bevonden kan worden/ dat bid ik U. E. Heeren mij 't zelve ten beste ende ten goede te houden/ en zo ik nog iets goets hebbe/ het welke tot profijt en voordeel van deze dijkazie ende gemeene lands welvaart zoude mogen dienen en trekken/ dat zelve wil ik tot allen tijde mede deelen/ en dienen met de gaven die mij den Heere gegeeven heeft."

("Herewith I will end my writings, and if I have missed ought herein which is not so well hit, as may be found during the diking, then I pray you noble sirs not to hold the same against me, and if I have yet something of value which might serve for the profit of this diking project and for the welfare of the commonwealth, the same I am ready to communicate at all times, and serve with the gifts that the Lord has given me.")

From: "Haarlemmer – meer – boek"
Jan Adriaanz Leeghwater, 1643

APPENDIX – FREQUENTLY USED SYMBOLS

Dimensions of the entities symbolized are denoted in terms of the symbols L for length, T for time, and 0 for dimensionless.

$c_j = D_j/K_j'$, vertical hydraulic resistance of layer "j"	T
$c' = c_1 + c_2/3$	T
$D_j = z_{j-1} - z_j$, thickness of layer "j"	L
$f = f(x,t)$, horizontal potential distribution (complex)	L
$g_j = g_j(z)$, vertical potential distribution in layer "j" (complex)	O
$h_j = g_j(z)f(x,t)$, hydraulic potential in layer "j" (complex)	L
K_j horizontal hydraulic conductivity of layer "j"	LT^{-1}
K_j' vertical hydraulic conductivity of layer "j"	LT^{-1}
m phase change per unit horizontal distance	L^{-1}
n change of natural logarithm of the amplitude per unit horizontal distance	L^{-1}
$p = n + im$, horizontal propagation parameter (complex)	L^{-1}
q_j vertical propagation parameter for layer "j"	L^{-1}
S_j' specific (elastic) storage coefficient of layer "j"	L^{-1}
$S_j = S_j'D_j$, storage coefficient of layer "j"	O
S_0 specific yield, or coefficient of storage at the water table	O
t time	T
T_W time lag constant of a well or piezometer	T
x horizontal distance coordinate	L
z vertical distance coordinate	L
z_0 average height of the water table	L
z_j height of top of layer "j+1"	L
ω angular frequency	T^{-1}

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